A unified WKB analysis of instabilities in magnetized Keplerian flows at low magnetic Prandtl number

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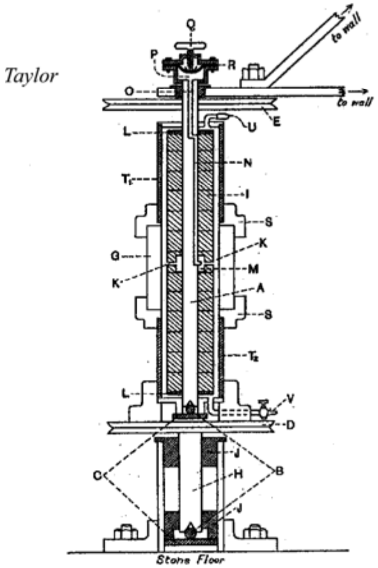
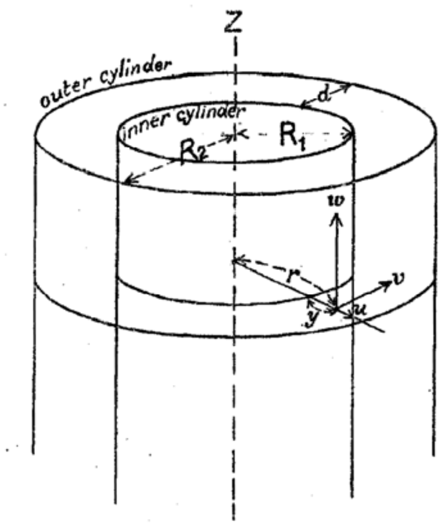
Outline

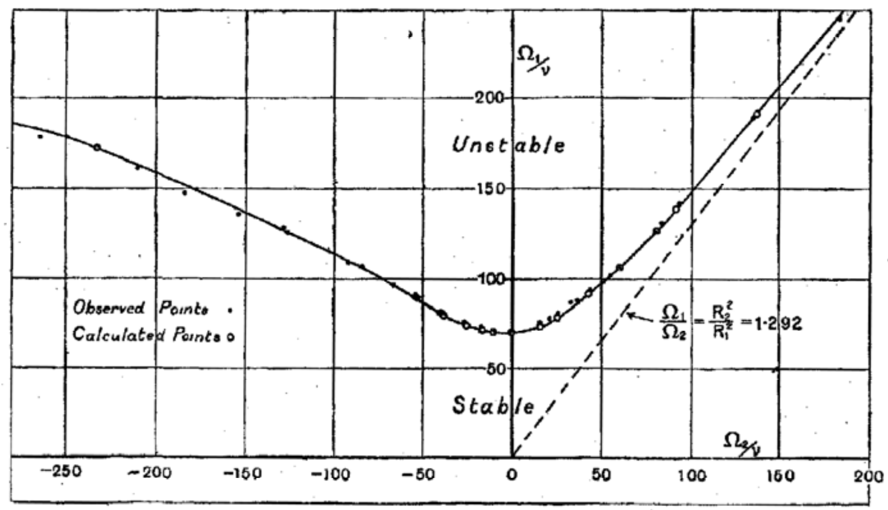
1. **Standard magnetorotational instability**
   a) Couette-Taylor flow
   b) Accretion disks
2. **Experiments with liquid metals**
   a) Axial field (SMRI)
   b) Helical field (HMRI)
3. **SMRI versus HMRI**
   a) Inductionless approximation and the Liu limit
   b) Deviation from the current-free azimuthal field may result in HMRI of Keplerian flows
4. **Azimuthal MRI versus Tayler instability**

O.N.K, F. Stefani Phys. Rev. Lett. **111**(6), 061103 (2013)

1923: Taylor’s linear stability diagram for the viscous rotating Couette flow







- Perfect agreement of the linear stability analysis with the experiment for moderate Reynolds numbers
- An asymptote in case of co-rotation is the Rayleigh criterion for the inviscid fluid

1917: Rayleigh's criterion for centrifugal stability

- Inviscid fluid, planar flow
- Axisymmetric perturbation

Stability: iff the angular momentum increases radially

$$\frac{1}{R^3}\frac{d}{dR}(\Omega R^2)^2 > 0$$

In terms of the hydrodynamic Rossby number

$$\mathrm{Ro} := \frac{R}{2\Omega}\frac{\partial \Omega}{\partial R} > -1$$

Corollary for the rotating Couette flow ($R_1 < R_2$):

$$\Omega_2 R_2^2 > \Omega_1 R_1^2$$

1959, Velikhov: new instability of the Rayleigh-stable magnetized Couette-Taylor flow

Standard magnetorotational instability (S)MRI

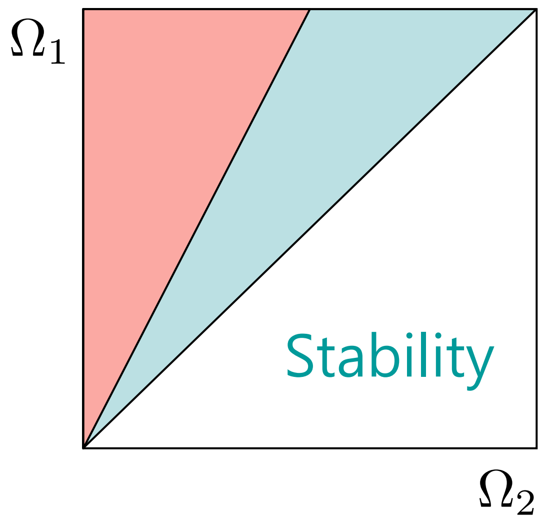
- CT-flow of inviscid fluid
- Perfect electrical conductor
- Uniform axial magnetic field
- Axisymmetric perturbation

Sufficient condition for stability

$$\frac{d\Omega^2}{dR} > 0 \quad or \quad \mathrm{Ro} > 0 \quad or \quad \Omega_2 > \Omega_1$$

E.P. Velikhov, Sov. Phys. JETP **36,** 995–998 (1959)

SMRI and centrifugal instability



Couette-Taylor flow

When magnetic field tends to zero, the SMRI domain does not shrink to the domain of centrifugal instability

SMRI	Centrifugal instability
Velikhov-Chandrasekhar	Rayleigh
$\Omega_2 < \Omega_1 \quad or \quad \mathrm{Ro} < 0$	$R_2^2\Omega_2 < R_1^2\Omega_1 \quad or \quad \mathrm{Ro} < -1$

S. Chandrasekhar, Proc. Natl. Acad. Sci. **46**, 253–257 (1960)

Physical reasons behind the paradox

Steven Balbus (Ann. Rev. Astron. Astroph. 2003):

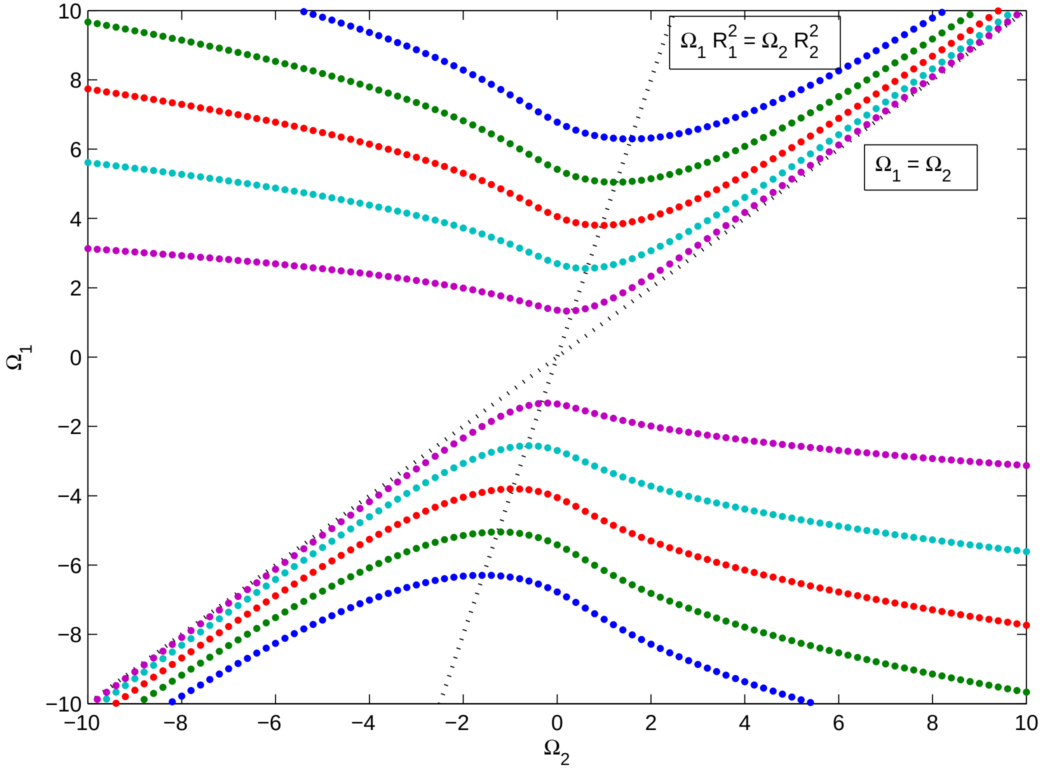
‘much of the fluid and stellar community was aware of the [MRI] and of its curious behavior of ostensibly changing the Rayleigh criterion discontinuously’

Existing physical explanation via Alfven's theorem:

In a fluid of zero resistivity the magnetic field lines are frozen-in to the fluid, independent on the strength of the magnetic field, thus angular velocity is preserved

In the ideal MHD the MRI threshold coincides with that of the solid body rotation in the limit of vanishing magnetic field only

O.N.K., Dmitry Pelinovsky, Guido Schneider, Physical Review E **84**, 065301(R) (2011)



In the viscous and resistive setting continuous transitions between the Rayleigh and Velikhov-Chandrasekhar lines are possible

A. P. Willis and C. F. Barenghi, A&A **388**, 688–691 (2002)

1991: Steven Balbus and John Hawley establish the astrophysical relevance of SMRI

Observations: flow in accretion disks is turbulent

The flow following Keplerian orbits

$$\Omega(R) = (GM)^{1/2}/R^{3/2} \;\Rightarrow\; d(R^2\Omega)/dR > 0,\; d\Omega/dR < 0$$

is centrifugally stable but SMRI-unstable:

$$-1 < \mathrm{Ro}_{Kep} = -\frac{3}{4} < 0$$

SMRI can trigger turbulence in Keplerian disks

S. A. Balbus and J. F. Hawley, The Astrophysical Journal **376**, 214–222 (1991)

Basic mechanism of SMRI in Keplerian disks

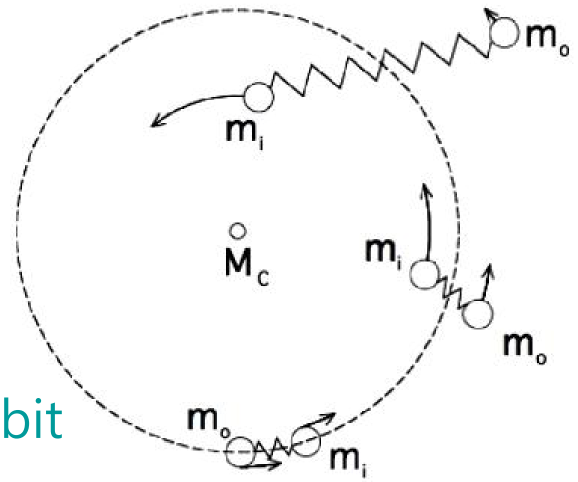
Fluid is a perfect conductor

Magnetic field lines frozen into the fluid

A line 'tethers' a couple of fluid particles

Inner mass retarded, moves to a lower orbit

Outer mass accelerated, moves to a higher orbit



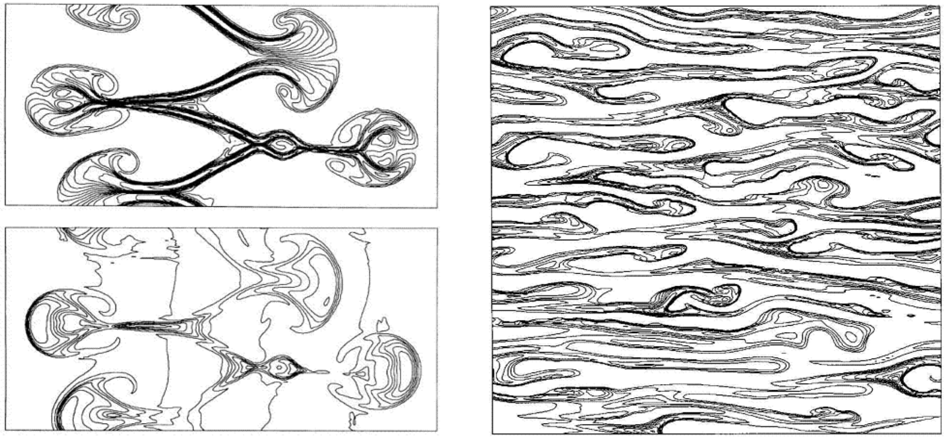


Figure 1: Evolution of the MRI as computed in (a) [19], (b) [20], with the horizontal and vertical coordinates corresponding to the radial and vertical directions in the disk.

Simulations

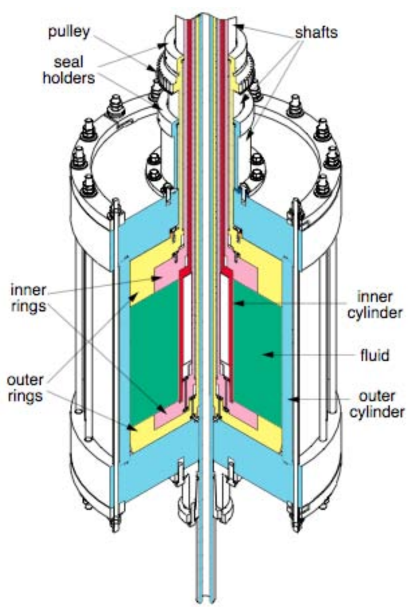
Balbus and Hawley

1991

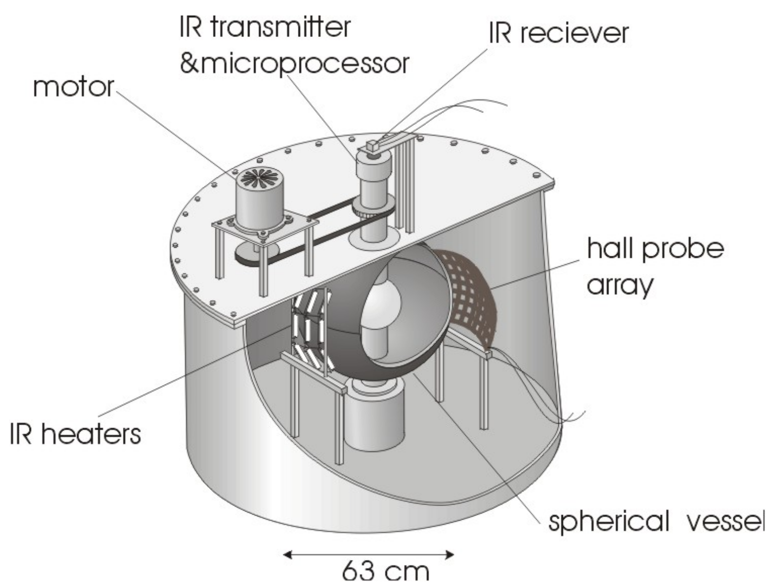
S. A. Balbus and J. F. Hawley, The Astrophysical Journal **376**, 214–222 (1991)

Detecting MRI in the Couette-Taylor cells

Despite the evident successes of the numerical modeling of MRI, the range of parameters that are typical in the astrophysical applications had still not been reached in the computer simulations. By this reason, increasing efforts are being taken during the last decade in order to detect the MRI in the laboratory.



Princeton



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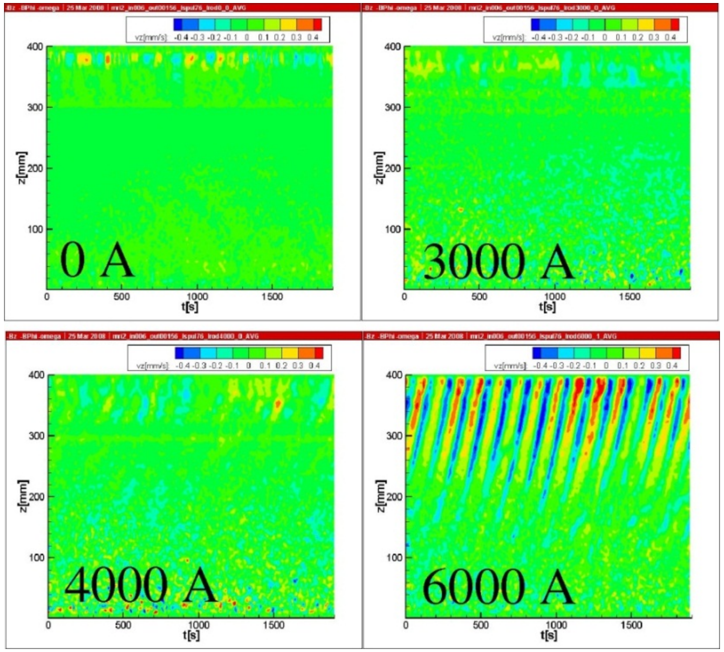
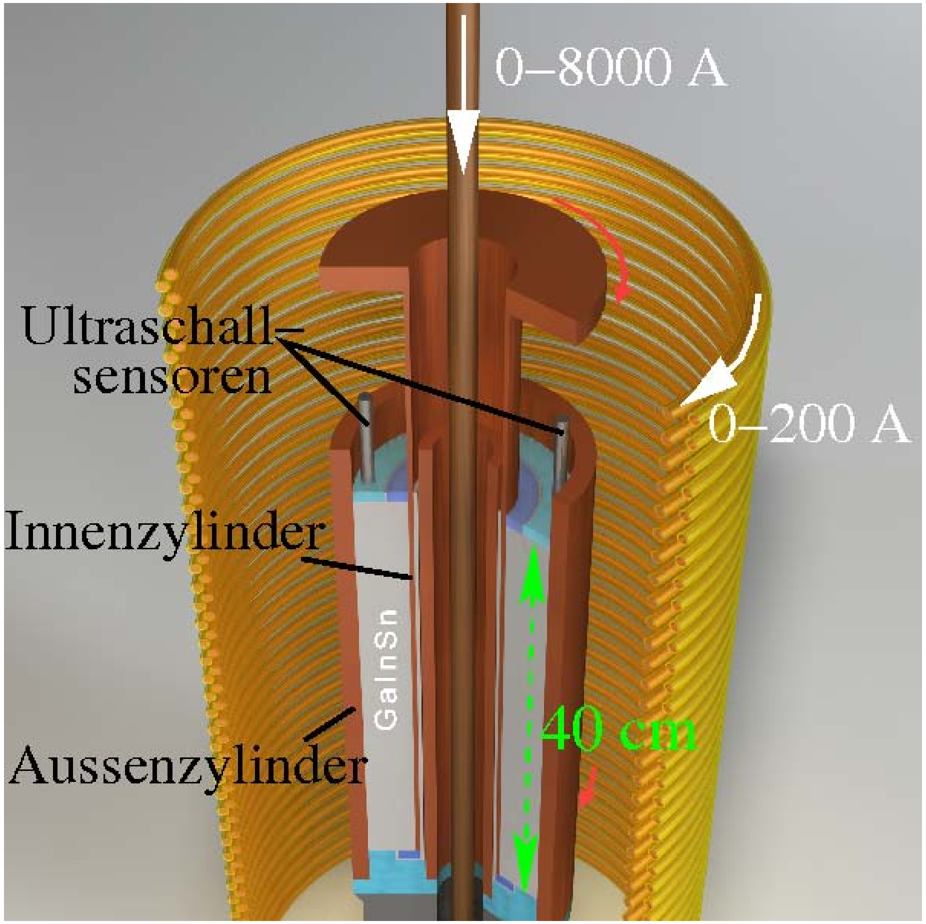
Helical version of MRI:
Observed in the PROMISE experiment in 2006





Stefani, F. et al., Phys. Rev. Lett. 97(2006), 184502
Rüdiger, G. et al., Astrophys.J. Lett. 649 (2006), L145
Stefani, F. et al., New J. Phys. 9 (2007), 295
Stefani, F. et al., Astron. Nachr. 329 (2008), 652-658
Priede, J., Gerbeth, G., Phys. Rev. E 79 (2009), 046310
Stefani et al., Phys. Rev. E 80 (2009), 066303

Helical MRI in the PROMISE experiment

Uniform axial magnetic field

The azimuthal magnetic field is produced by the axial current:

$$B_\phi(R) \propto \frac{1}{R}$$

In these circumstances there is no HMRI if **(Liu et al. 2006)**

$$\mathrm{Ro} > \mathrm{Ro}_{Liu} := 2 - 2\sqrt{2} \approx -0.828$$

The Keplerian value Ro=-0.75 is above the **Liu limit**

This prohibits HMRI of Keplerian flows

W. Liu, J. Goodman, I. Herron, H. Ji, Phys. Rev. E **74**, 056302 (2006)

Axisymmetric Standard and Helical MRI

Standard MRI (SMRI) – axial magnetic field perpendicular to the disk

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Standard MRI – theoretically destabilizes Keplerian flows, not yet observed in the laboratory **(Critical Re~10^6)**

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Helical MRI – already detected in the experiment **(Critical Re~1000)**

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if $B_\phi^0(R) \sim R^{-1}$ – no destabilization of Keplerian flows even in theory

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At what $B_\phi^0(R)$ can HMRI destabilize the Keplerian flows?

Mathematical setting

Navier-Stokes equation for the fluid velocity $\boldsymbol{u}$

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} - \frac{1}{\mu_0 \rho} \boldsymbol{B} \cdot \nabla \boldsymbol{B} + \frac{1}{\rho} \nabla P - \nu \nabla^2 \boldsymbol{u} = 0$$

Induction equation for the magnetic field $\boldsymbol{B}$

$$\frac{\partial \boldsymbol{B}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{B} - \boldsymbol{B} \cdot \nabla \boldsymbol{u} - \eta \nabla^2 \boldsymbol{B} = 0$$

Mass continuity for incompressible flows and the solenoidal condition

$$\nabla \cdot \boldsymbol{u} = 0, \quad \nabla \cdot \boldsymbol{B} = 0$$

p : pressure, $\rho = const$: density, $\nu = const$: kinematic viscosity

$\eta = (\mu_0 \sigma)^{-1}$: magnetic diffusivity, σ : conductivity of the fluid

μ_0 : magnetic permeability of free space

O.N.K., F. Stefani The Astrophysical Journal **712**, 52–68 (2010)

Perturbation of the steady state

Steady state: **A magnetized Taylor-Couette flow**

$$\boldsymbol{u}_0(R) = R\Omega(R)\,\boldsymbol{e}_\phi, \quad p = p_0(R), \quad \boldsymbol{B}_0(R) = B_\phi^0(R)\boldsymbol{e}_\phi + B_z^0\boldsymbol{e}_z, \quad R\Omega^2 = \frac{1}{\rho}\frac{\partial p_0}{\partial R}$$

Hydrodynamic (Ro) and magnetic (Rb) **Rossby numbers:**

$$\mathrm{Ro} = \frac{R}{2\Omega}\frac{\partial \Omega}{\partial R} \qquad \mathrm{Rb} = \frac{R}{2\,(B_\phi/R)}\frac{\partial\,(B_\phi/R)}{\partial R}$$

For the current-free azimuthal field $B_\phi(R) \propto \dfrac{1}{R} \Rightarrow \mathrm{Rb} = -1$

General perturbation: $\boldsymbol{u} = \boldsymbol{u}_0 + \boldsymbol{u}', \quad p = p_0 + p', \quad \boldsymbol{B} = \boldsymbol{B}_0 + \boldsymbol{B}'$

Gradients:

$$\mathcal{U}(R) = \nabla \boldsymbol{u}_0 = \Omega \begin{pmatrix} 0 & -1 & 0 \\ 1+2\mathrm{Ro} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{B}(R) = \nabla \boldsymbol{B}_0 = \frac{B_\phi^0}{R}\begin{pmatrix} 0 & -1 & 0 \\ 1+2\mathrm{Rb} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

O.N.K., F. Stefani Phys. Rev. Lett. **111**(6), 061103 (2013)

Linearized equations

$$(\partial_t + \mathcal{U} + \boldsymbol{u}_0 \cdot \nabla)\boldsymbol{u}' + \frac{1}{\rho}\nabla p' + \frac{1}{\rho\mu_0}\boldsymbol{B}_0 \times (\nabla \times \boldsymbol{B}') + \frac{1}{\rho\mu_0}\boldsymbol{B}' \times (\nabla \times \boldsymbol{B}_0) = \nu\nabla^2\boldsymbol{u}'$$

$$(\partial_t - \mathcal{U} + \boldsymbol{u}_0 \cdot \nabla)\boldsymbol{B}' + (\mathcal{B} - \boldsymbol{B}_0 \cdot \nabla)\boldsymbol{u}' = \eta\nabla^2\boldsymbol{B}'$$

$$\nabla \cdot \boldsymbol{u}' = 0, \quad \nabla \cdot \boldsymbol{B}' = 0$$

Geometrical optics approximation $(0 < \epsilon \ll 1)$

$$\boldsymbol{u}'(\boldsymbol{x}, t, \epsilon) = e^{i\Phi(\boldsymbol{x},t)/\epsilon}\left(\boldsymbol{u}^{(0)}(\boldsymbol{x}, t) + \epsilon\boldsymbol{u}^{(1)}(\boldsymbol{x}, t)\right) + \epsilon\boldsymbol{u}^{r}(\boldsymbol{x}, t)$$

$$\boldsymbol{B}'(\boldsymbol{x}, t, \epsilon) = e^{i\Phi(\boldsymbol{x},t)/\epsilon}\left(\boldsymbol{B}^{(0)}(\boldsymbol{x}, t) + \epsilon\boldsymbol{B}^{(1)}(\boldsymbol{x}, t)\right) + \epsilon\boldsymbol{B}^{r}(\boldsymbol{x}, t)$$

$$p'(\boldsymbol{x}, t, \epsilon) = e^{i\Phi(\boldsymbol{x},t)/\epsilon}\left(p^{(0)}(\boldsymbol{x}, t) + \epsilon p^{(1)}(\boldsymbol{x}, t)\right) + \epsilon p^{r}(\boldsymbol{x}, t)$$

S. Friedlander, M. M. Vishik Chaos **5**, 416–423 (1995)

Assuming $\frac{D}{Dt} := \partial_t + \boldsymbol{u}_0 \cdot \nabla, \quad \nu = \epsilon^2 \tilde{\nu}, \quad \eta = \epsilon^2 \tilde{\eta}$ and

collecting terms at ϵ^{-1} and ϵ^0, we get four equations

$$\frac{D\Phi}{Dt}\boldsymbol{u}^{(0)} + \frac{p^{(0)}}{\rho}\nabla\Phi - \frac{1}{\rho\mu_0}\boldsymbol{B}^{(0)}(\boldsymbol{B}_0 \cdot \nabla\Phi) + \frac{1}{\rho\mu_0}\nabla\Phi(\boldsymbol{B}_0 \cdot \boldsymbol{B}^{(0)}) = 0$$

$$\frac{D\Phi}{Dt}\boldsymbol{B}^{(0)} - (\boldsymbol{B}_0 \cdot \nabla\Phi)\boldsymbol{u}^{(0)} = 0$$

$$\left(\frac{D}{Dt} + \tilde{\nu}(\nabla\Phi)^2 + \mathcal{U}\right)\boldsymbol{u}^{(0)} + \frac{1}{\rho}\nabla p^{(0)} + i\frac{D\Phi}{Dt}\boldsymbol{u}^{(1)} + i\frac{p^{(1)}}{\rho}\nabla\Phi$$
$$+\frac{1}{\rho\mu_0}\boldsymbol{B}_0 \times \nabla \times \boldsymbol{B}^{(0)} - i\frac{1}{\rho\mu_0}\boldsymbol{B}_0 \times \boldsymbol{B}^{(1)} \times \nabla\Phi + \frac{1}{\rho\mu_0}\boldsymbol{B}^{(0)} \times \nabla \times \boldsymbol{B}_0 = 0$$

$$i\frac{D\Phi}{Dt}\boldsymbol{B}^{(1)} + \left(\frac{D}{Dt} + \tilde{\eta}(\nabla\Phi)^2 - \mathcal{U}\right)\boldsymbol{B}^{(0)} + (\mathcal{B} - \boldsymbol{B}_0 \cdot \nabla)\boldsymbol{u}^{(0)} - i(\boldsymbol{B}_0 \cdot \nabla\Phi)\boldsymbol{u}^{(1)} = 0$$

O.N.K., F. Stefani, Y. Fukumoto Journal of Fluid Mechanics, (2014) subm. arXiv:1401.8276

The first two equations yield

$$p^{(0)} = -\frac{1}{\mu_0}(\boldsymbol{B}_0 \cdot \boldsymbol{B}^{(0)}), \quad \frac{D\Phi}{Dt} = 0, \quad \boldsymbol{B}_0 \cdot \nabla\Phi = 0$$

Then, the last two equations take the form

$$\left(\frac{D}{Dt} + \tilde{\nu}(\nabla\Phi)^2 + \mathcal{U}\right)\boldsymbol{u}^{(0)} - \frac{1}{\rho\mu_0}(\mathcal{B} + \boldsymbol{B}_0 \cdot \nabla)\boldsymbol{B}^{(0)}$$

$$= \frac{\nabla\Phi}{|\nabla\Phi|^2} \cdot \left[\mathcal{U}\boldsymbol{u}^{(0)} - \frac{1}{\rho\mu_0}(\mathcal{B} + \boldsymbol{B}_0 \cdot \nabla)\boldsymbol{B}^{(0)}\right]\nabla\Phi - \frac{\nabla\Phi}{|\nabla\Phi|^2}\frac{D\nabla\Phi}{Dt} \cdot \boldsymbol{u}^{(0)}$$

$$\left(\frac{D}{Dt} + \tilde{\eta}(\nabla\Phi)^2 - \mathcal{U}\right)\boldsymbol{B}^{(0)} + (\mathcal{B} - \boldsymbol{B}_0 \cdot \nabla)\boldsymbol{u}^{(0)} = 0$$

O.N.K., F. Stefani, Y. Fukumoto Journal of Fluid Mechanics, (2014) subm. arXiv:1401.8276

Taking the gradient of the equation $\frac{D\Phi}{Dt} = 0$

$$\nabla \partial_t \Phi + \nabla(\boldsymbol{u}_0 \cdot \nabla)\Phi = \partial_t \nabla \Phi + (\boldsymbol{u}_0 \cdot \nabla)\nabla\Phi + \mathcal{U}^T \nabla \Phi = \frac{D}{Dt}\nabla\Phi + \mathcal{U}^T \nabla\Phi = 0$$

yields the phase equation

$$\frac{D\boldsymbol{k}}{Dt} = -\mathcal{U}^T \boldsymbol{k}, \quad \boldsymbol{k} = \nabla\Phi$$

In cylindrical coordinates

$$\dot{k}_R = -R\partial_R \Omega k_\phi, \quad \dot{k}_\phi = 0, \quad \dot{k}_z = 0$$

Bounded and asymptotically non-decaying solutions:

$$k_\phi \equiv 0, \quad k_R = const, \quad k_z = const$$

$$\alpha = k_z |\boldsymbol{k}|^{-1}, \quad |\boldsymbol{k}|^2 = k_R^2 + k_z^2$$

B. Eckhardt, D. Yao Chaos, Solitons, and Fractals **5**, 2073-2088 (1995)

Amplitude equations (PDEs)

$$\frac{D\boldsymbol{u}^{(0)}}{Dt}=-\left(\mathcal{I}-2\frac{\boldsymbol{k}\boldsymbol{k}^T}{|\boldsymbol{k}|^2}\right)\mathcal{U}\boldsymbol{u}^{(0)}-\tilde{\nu}|\boldsymbol{k}|^2\boldsymbol{u}^{(0)}+\frac{1}{\rho\mu_0}\left(\mathcal{I}-\frac{\boldsymbol{k}\boldsymbol{k}^T}{|\boldsymbol{k}|^2}\right)(\mathcal{B}+\boldsymbol{B}_0\cdot\nabla)\,\boldsymbol{B}^{(0)}$$

$$\frac{D\boldsymbol{B}^{(0)}}{Dt}=\mathcal{U}\boldsymbol{B}^{(0)}-\tilde{\eta}|\boldsymbol{k}|^2\boldsymbol{B}^{(0)}-(\mathcal{B}-\boldsymbol{B}_0\cdot\nabla)\boldsymbol{u}^{(0)}$$

In cylindrical coordinates

$$\partial_t u_R^{(0)}+\Omega\left(\partial_\phi u_R^{(0)}-u_\phi^{(0)}\right)+\tilde{\nu}|\boldsymbol{k}|^2u_R^{(0)}+\Omega u_\phi^{(0)}-2\Omega u_\phi^{(0)}\alpha^2+\frac{B_\phi^0}{\rho\mu_0 R}2\alpha^2B_\phi^{(0)}-\alpha\frac{B_\phi^0}{R}\frac{\alpha\partial_\phi B_R^{(0)}-\sqrt{1-\alpha^2}\partial_\phi B_z^{(0)}}{\rho\mu_0}-\alpha B_z^0\frac{\alpha\partial_z B_R^{(0)}-\sqrt{1-\alpha^2}\partial_z B_z^{(0)}}{\rho\mu_0}=0$$

$$\partial_t u_\phi^{(0)}+\Omega\left(\partial_\phi u_\phi^{(0)}+u_R^{(0)}\right)+(\Omega+2\Omega\mathrm{Ro})u_R^{(0)}+\tilde{\nu}|\boldsymbol{k}|^2u_\phi^{(0)}-\frac{1}{\rho\mu_0}\frac{B_\phi^0}{R}\partial_\phi B_\phi^{(0)}-\frac{B_z^0\partial_z B_\phi^{(0)}}{\rho\mu_0}-\frac{2}{\rho\mu_0}\frac{B_\phi^0}{R}(1+\mathrm{Rb})B_R^{(0)}=0$$

$$\partial_t u_z^{(0)}+\Omega\partial_\phi u_z^{(0)}+2\sqrt{1-\alpha^2}\alpha\Omega u_\phi^{(0)}+\tilde{\nu}|\boldsymbol{k}|^2u_z^{(0)}+\frac{B_\phi^0(\alpha^2-1)\partial_\phi B_z^{(0)}}{\rho\mu_0 R}+\frac{\alpha\sqrt{1-\alpha^2}B_\phi^0(\partial_\phi B_R^{(0)}-2B_\phi^{(0)})}{\rho\mu_0 R}+B_z^0\frac{\sqrt{1-\alpha^2}\alpha\partial_z B_R^{(0)}-(1-\alpha^2)\partial_z B_z^{(0)}}{\rho\mu_0}=0$$

$$\partial_t B_R^{(0)}+\Omega\left(\partial_\phi B_R^{(0)}-B_\phi^{(0)}\right)+\Omega B_\phi^{(0)}+\tilde{\eta}|\boldsymbol{k}|^2B_R^{(0)}-B_z^0\partial_z u_R^{(0)}-\frac{B_\phi^0}{R}\partial_\phi u_R^{(0)}=0$$

$$\partial_t B_z^{(0)}+\Omega\partial_\phi B_z^{(0)}+\tilde{\eta}|\boldsymbol{k}|^2B_z^{(0)}-B_z^0\partial_z u_z^{(0)}-\frac{B_\phi^0}{R}\partial_\phi u_z^{(0)}=0$$

$$\partial_t B_\phi^{(0)}+\Omega\left(\partial_\phi B_\phi^{(0)}+B_R^{(0)}\right)-(\Omega+2\Omega\mathrm{Ro})B_R^{(0)}+\tilde{\eta}|\boldsymbol{k}|^2B_\phi^{(0)}-B_z^0\partial_z u_\phi^{(0)}+2\mathrm{Rb}\frac{B_\phi^0}{R}u_R^{(0)}-\frac{B_\phi^0}{R}\partial_\phi u_\phi^{(0)}=0$$

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Axi-symmetric solutions

$$\sim e^{\gamma t + i k_z z}$$

Reduction to four equations

$$(\gamma + \widetilde{\nu}|\boldsymbol{k}|^2)u_R^{(0)} - 2\alpha^2\Omega u_\phi^{(0)} + 2\alpha^2\frac{B_\phi^0}{\rho\mu_0 R}B_\phi^{(0)} - \frac{ik_z B_z^0}{\rho\mu_0}B_R^{(0)} = 0$$

$$(\gamma + \widetilde{\nu}|\boldsymbol{k}|^2)u_\phi^{(0)} + 2\Omega(1+\mathrm{Ro})u_R^{(0)} - \frac{2}{\rho\mu_0}\frac{B_\phi^0}{R}(1+\mathrm{Rb})B_R^{(0)} - \frac{ik_z B_z^0}{\rho\mu_0}B_\phi^{(0)} = 0$$

$$(\gamma + \widetilde{\eta}|\boldsymbol{k}|^2)B_R^{(0)} - ik_z B_z^0 u_R^{(0)} = 0$$

$$(\gamma + \widetilde{\eta}|\boldsymbol{k}|^2)B_\phi^{(0)} - 2\Omega\mathrm{Ro}B_R^{(0)} + 2\mathrm{Rb}\frac{B_\phi^0}{R}u_R^{(0)} - ik_z B_z^0 u_\phi^{(0)} = 0$$

O.N.K., F. Stefani, Y. Fukumoto Journal of Fluid Mechanics, (2014) subm. arXiv:1401.8276

Viscous, resisitve, and Alfven frequencies:

$$\omega_\nu = \widetilde{\nu}|\boldsymbol{k}|^2, \quad \omega_\eta = \widetilde{\eta}|\boldsymbol{k}|^2, \quad \omega_A^2 = \frac{(k_z B_z^0)^2}{\rho\mu_0}, \quad \omega_{A_\phi}^2 = \frac{(B_\phi^0)^2}{\rho\mu_0 R^2}$$

Dispersion relation

$$H = \begin{pmatrix} -\omega_\nu & 2\alpha^2\Omega & i\frac{\omega_A}{\sqrt{\rho\mu_0}} & -\frac{2\omega_{A_\phi}\alpha^2}{\sqrt{\rho\mu_0}} \\ -2\Omega(1+\mathrm{Ro}) & -\omega_\nu & \frac{2\omega_{A_\phi}}{\sqrt{\rho\mu_0}}(1+\mathrm{Rb}) & i\frac{\omega_A}{\sqrt{\rho\mu_0}} \\ i\omega_A\sqrt{\rho\mu_0} & 0 & -\omega_\eta & 0 \\ -2\omega_{A_\phi}\mathrm{Rb}\sqrt{\rho\mu_0} & i\omega_A\sqrt{\rho\mu_0} & 2\Omega\mathrm{Ro} & -\omega_\eta \end{pmatrix}$$

Complex polynomial: $p(\gamma) := \det(H - \gamma E) = 0$

$$p(\gamma) = (a_0 + ib_0)\gamma^4 + (a_1 + ib_1)\gamma^3 + (a_2 + ib_2)\gamma^2 + (a_3 + ib_3)\gamma + a_4 + ib_4$$

O.N.K., F. Stefani Phys. Rev. Lett. **111**(6), 061103 (2013)

Bilharz stability criterion for complex polynomials

Bilharz matrix

$$
\mathbf{B} = \begin{pmatrix}
a_4 & -b_4 & 0 & 0 & 0 & 0 & 0 & 0 \\
b_3 & a_3 & a_4 & -b_4 & 0 & 0 & 0 & 0 \\
-a_2 & b_2 & b_3 & a_3 & a_4 & -b_4 & 0 & 0 \\
-b_1 & -a_1 & -a_2 & b_2 & b_3 & a_3 & a_4 & -b_4 \\
a_0 & -b_0 & -b_1 & -a_1 & -a_2 & b_2 & b_3 & a_3 \\
0 & 0 & a_0 & -b_0 & -b_1 & -a_1 & -a_2 & b_2 \\
0 & 0 & 0 & 0 & a_0 & -b_0 & -b_1 & -a_1 \\
0 & 0 & 0 & 0 & 0 & 0 & a_0 & -b_0
\end{pmatrix}
$$

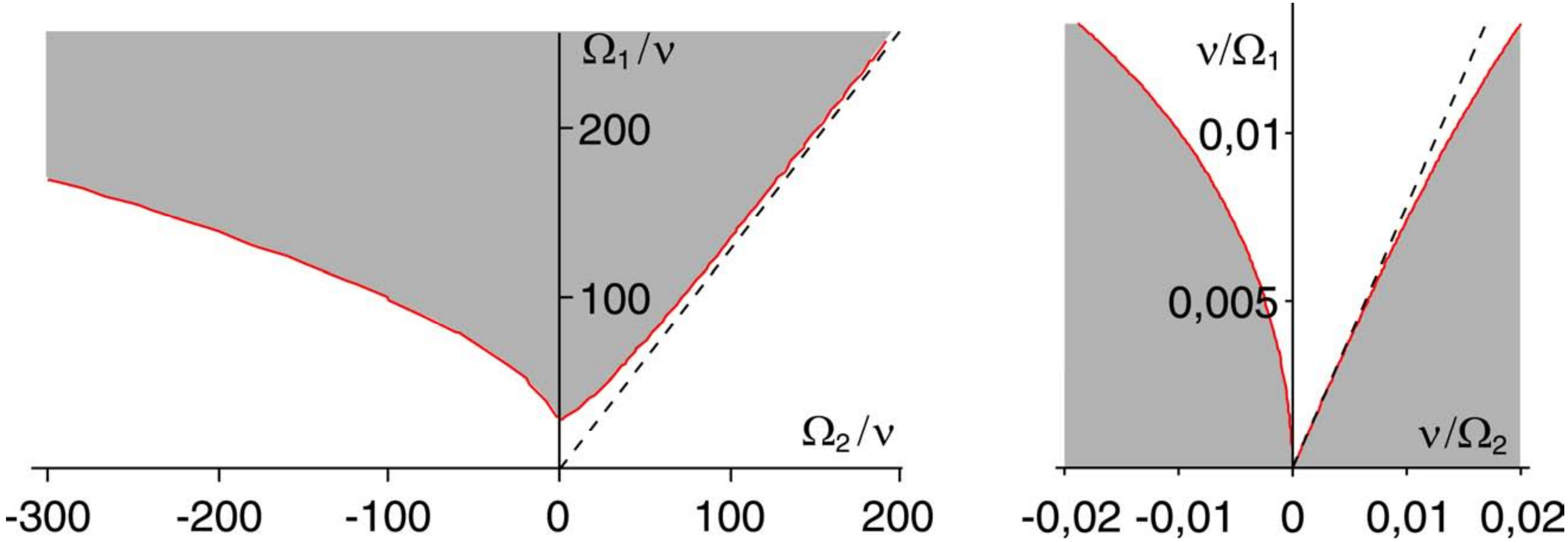
$$
\det \begin{pmatrix} a_4 & -b_4 \\ b_3 & a_3 \end{pmatrix} > 0, \quad \ldots, \quad \det \mathbf{B} > 0
$$

H. Bilharz ZAMM 24, (1944) 77-82

WKB stability boundary for non-magnetic CT-flow

Onset of SMRI: $$\mathrm{Ro} = -\frac{\omega_\eta^2}{\omega_A^2+\omega_\eta^2} - \frac{(\omega_A^2+\omega_\nu\omega_\eta)^2}{4\Omega_0^2\alpha^2(\omega_A^2+\omega_\eta^2)}$$

$$\omega_A = 0: \quad \mathrm{Ro} = -1 - \frac{\omega_\nu^2}{4\Omega_0^2\alpha^2}$$



B. Eckhardt, D. Yao Chaos, Solitons & Fractals 5, (1995) 2073

The paradox of Velikhov and Chandrasekhar

What laws of differential rotation are susceptible to SMRI?

$$\mathrm{Ro} < -\frac{\omega_\eta^2}{\omega_A^2+\omega_\eta^2} - \frac{(\omega_A^2+\omega_\nu\omega_\eta)^2}{4\Omega_0^2\alpha^2(\omega_A^2+\omega_\eta^2)}$$

Non-magnetic case (Rayleigh, Proc. R. Soc. Lond. A 1917)

$\omega_A = 0,\ \omega_\nu = 0$

$$\mathrm{Ro} < -1$$

Ideal MHD case (Velikhov, JETP 1959; Chandrasekhar, PNAS 1960)

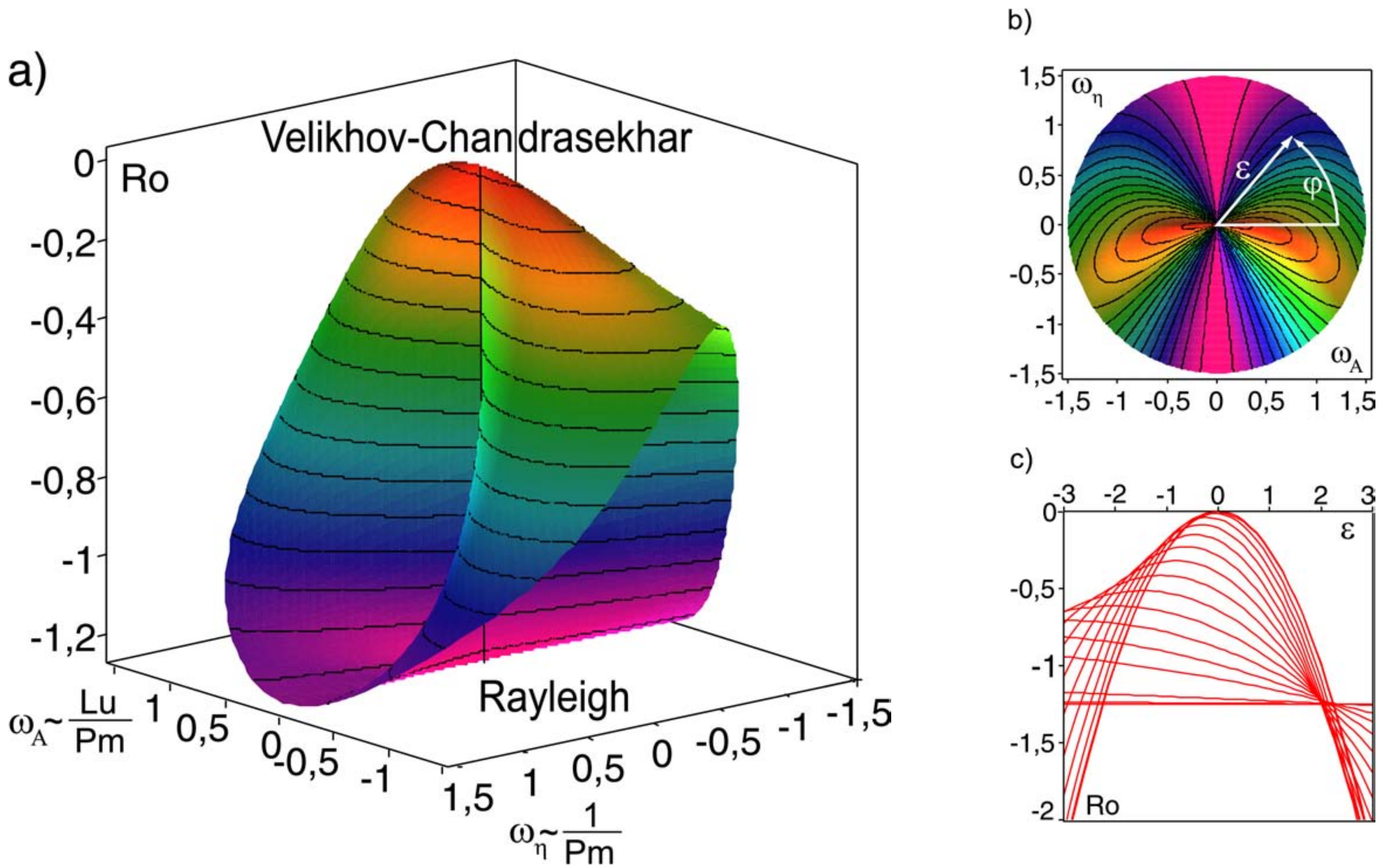
$\omega_\eta = 0,\ \omega_\nu = 0$

$$\mathrm{Ro} < -\frac{\omega_A^2}{4\Omega_0^2\alpha^2}$$

However, in the limit of vanishing magnetic field $\mathrm{Ro} < 0$!

O.N.K., F. Stefani Acta Applicandae Mathematicae **120**, 177-198 (2012)

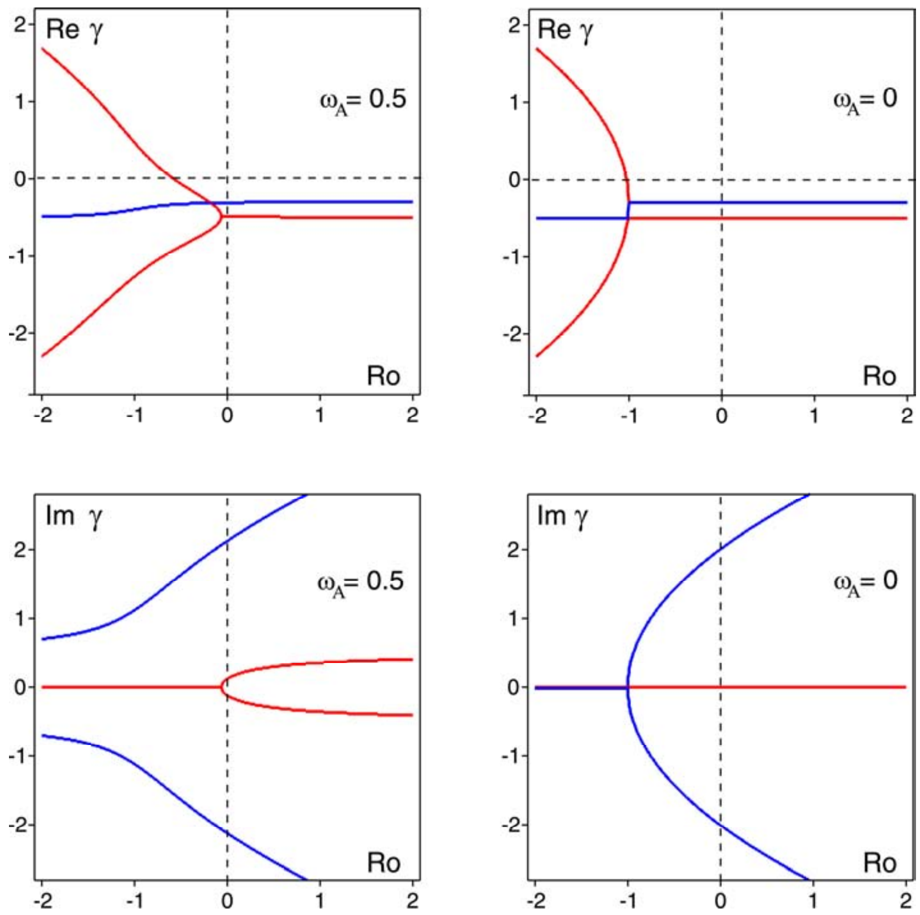
Stability boundary's singularities explain the paradox



O.N.K., F. Stefani Physical Review E **84**, 036304 (2011)

Transition from Magneto-Coriolis waves to inertial waves

$\Omega_0=1,\ \alpha=1,\ \omega_\nu=0.3,\ \omega_\eta=0.5$



$\omega_A = 0$ – Inertial wave:

$$\gamma_{1,2} = -\omega_\nu \pm i2\alpha\Omega_0\sqrt{\mathrm{Ro}+1}$$

$$\gamma_{3,4} = -\omega_\eta$$

Critical Rossby number:

$$\mathrm{Ro} = -1 - \frac{\omega_\nu^2}{4\alpha^2\Omega_0^2}$$

$\omega_A \neq 0$ – MC wave, undamped:

$$\gamma^2 = -2\Omega_0^2\alpha^2(1+\mathrm{Ro}) - \omega_A^2$$

$$\pm 2\Omega_0\alpha\sqrt{\Omega_0^2\alpha^2(1+\mathrm{Ro})^2 + \omega_A^2}$$

O.N.K., F. Stefani Acta Applicandae Mathematicae **120**, 177-198 (2012)

HZDR

Dimensionless dispersion relation:

$$\det\left(\widetilde{H}-\lambda E\right)=0$$

λ is a complex growth rate measured in units of $\alpha\Omega$

$$\widetilde{H}=\begin{pmatrix} -\frac{1}{\mathrm{Re}} & 2\alpha & \frac{i\mathrm{Ha}}{\sqrt{\mathrm{ReRm}}} & \frac{-2\alpha\beta\mathrm{Ha}}{\sqrt{\mathrm{ReRm}}} \\ \frac{-2(1+\mathrm{Ro})}{\alpha} & -\frac{1}{\mathrm{Re}} & \frac{2\beta\mathrm{Ha}(1+\mathrm{Rb})}{\alpha\sqrt{\mathrm{ReRm}}} & \frac{i\mathrm{Ha}}{\sqrt{\mathrm{ReRm}}} \\ \frac{i\mathrm{Ha}}{\sqrt{\mathrm{ReRm}}} & 0 & -\frac{1}{\mathrm{Rm}} & 0 \\ \frac{-2\beta\mathrm{HaRb}}{\alpha\sqrt{\mathrm{ReRm}}} & \frac{i\mathrm{Ha}}{\sqrt{\mathrm{ReRm}}} & \frac{2\mathrm{Ro}}{\alpha} & -\frac{1}{\mathrm{Rm}} \end{pmatrix}$$

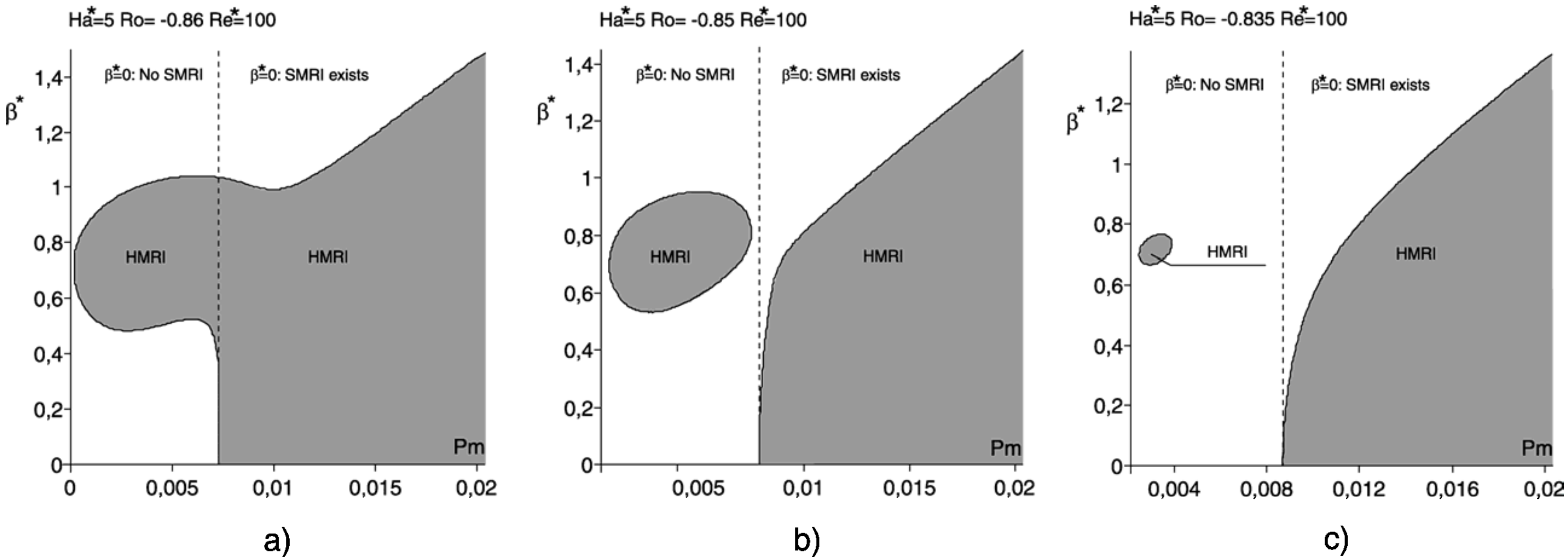
Parameters

$$\beta=\alpha\frac{\omega_{A_\phi}}{\omega_A},\quad \mathrm{Re}=\alpha\frac{\Omega}{\omega_\nu},\quad \mathrm{Ha}=\frac{\omega_A}{\sqrt{\omega_\nu\omega_\eta}},\quad \mathrm{Pm}=\frac{\omega_\nu}{\omega_\eta},\quad \mathrm{Rm}=\mathrm{RePm}$$

O.N.K., F. Stefani Phys. Rev. Lett. **111**(6), 061103 (2013)

HZDR

Helical magnetorotational instability (HMRI) at Rb=-1



Dashed line: SMRI – threshold for β = 0

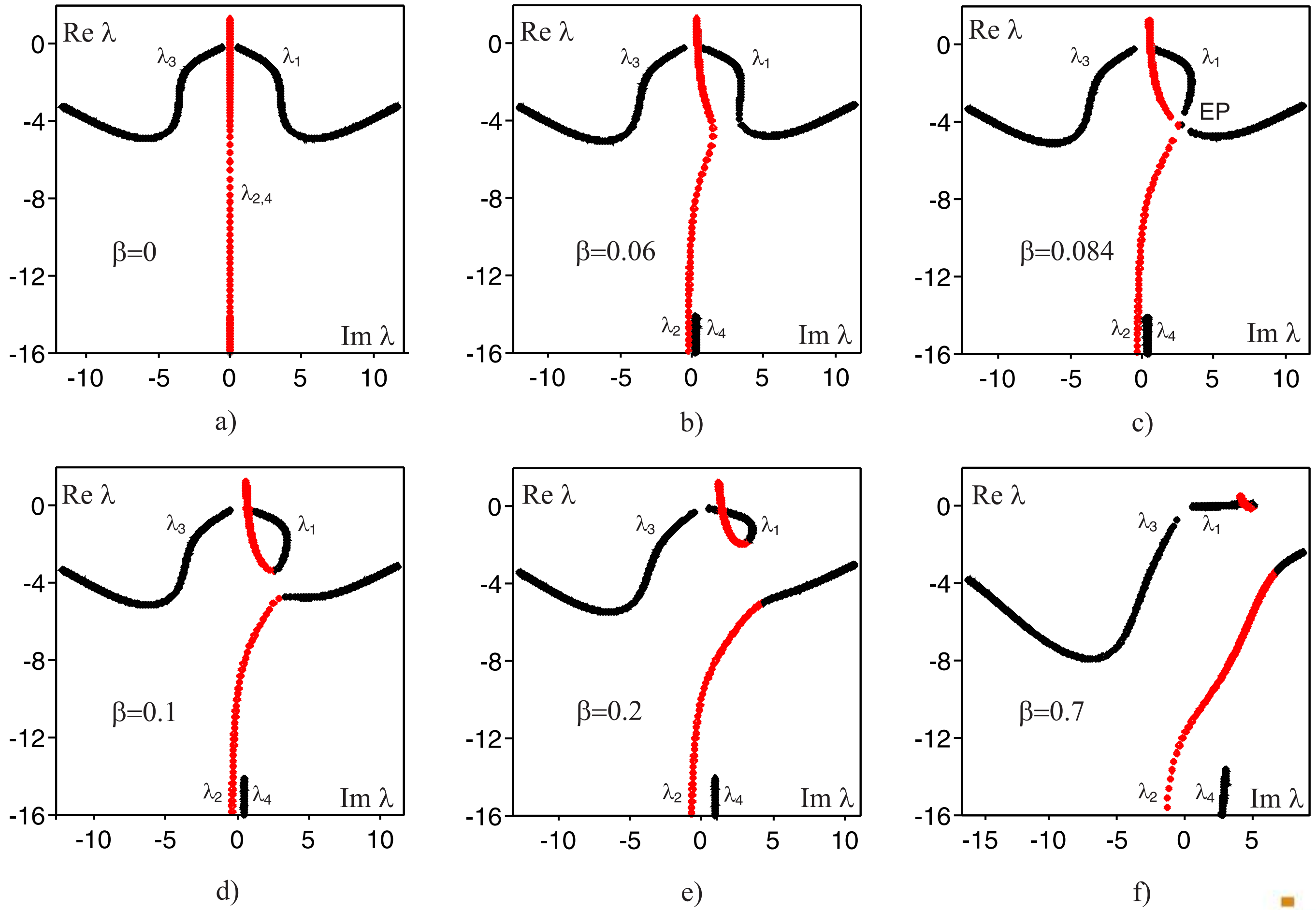
$$\mathrm{Pm} = \mathrm{Pm}^c := -\frac{\left(1+\mathrm{Ha}^2\right)^2 + 4\mathrm{Re}^2(1+\mathrm{Ro})}{4\mathrm{Ro}\mathrm{Re}^2\mathrm{Ha}^2}$$

Left to the dashed line – a (semi-) „island" of the essential HMRI

Right to the dashed line – a „continent" of the helically modified SMRI

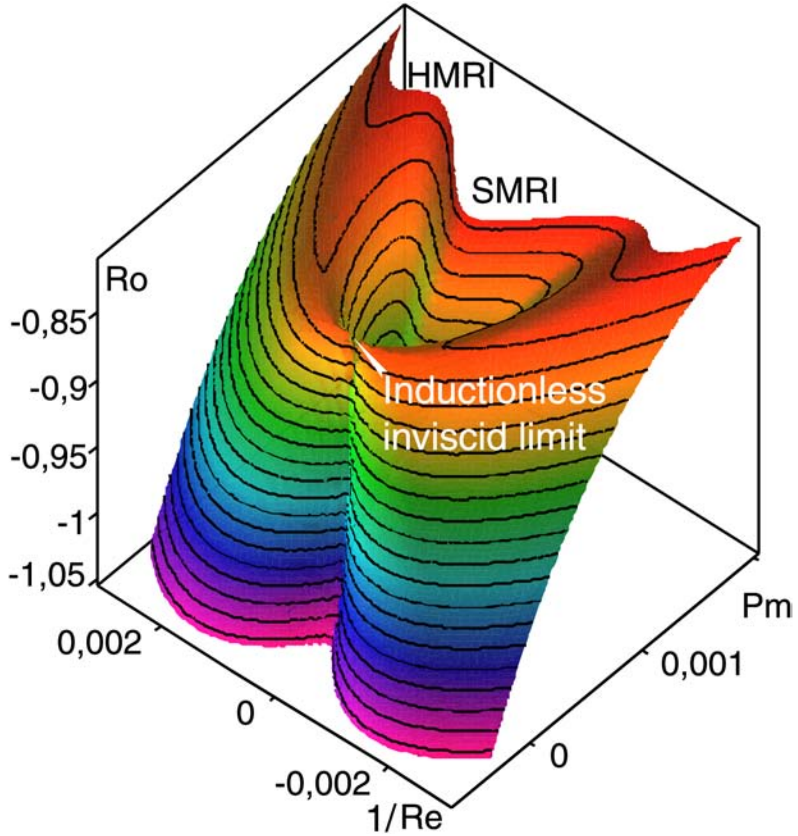
O.N.K., F. Stefani The Astrophysical Journal 712, (2010) 52–68

HMRI as a result of the interaction of slow MC-wave (SMRI) and inertial wave as Pm increases from 0 to 0.011 at Ha=5, Re=100, Ro=-0.85, Rb=-1



O.N.K., F. Stefani The Astrophysical Journal 712, (2010) 52–68

Instability threshold at low Pm



$$\mathrm{Rb} = -1, \quad \mathrm{Ha} = 15, \quad \beta = 0.7$$

O.N.K., F. Stefani Physical Review E **84**, 036304 (2011)

The inductionless approximation

In contrast to SMRI, Helical MRI exists at $\mathrm{Pm} = 0$

In weakly conducting media, i.e. liquid metals, $\mathrm{Pm} \ll 1$

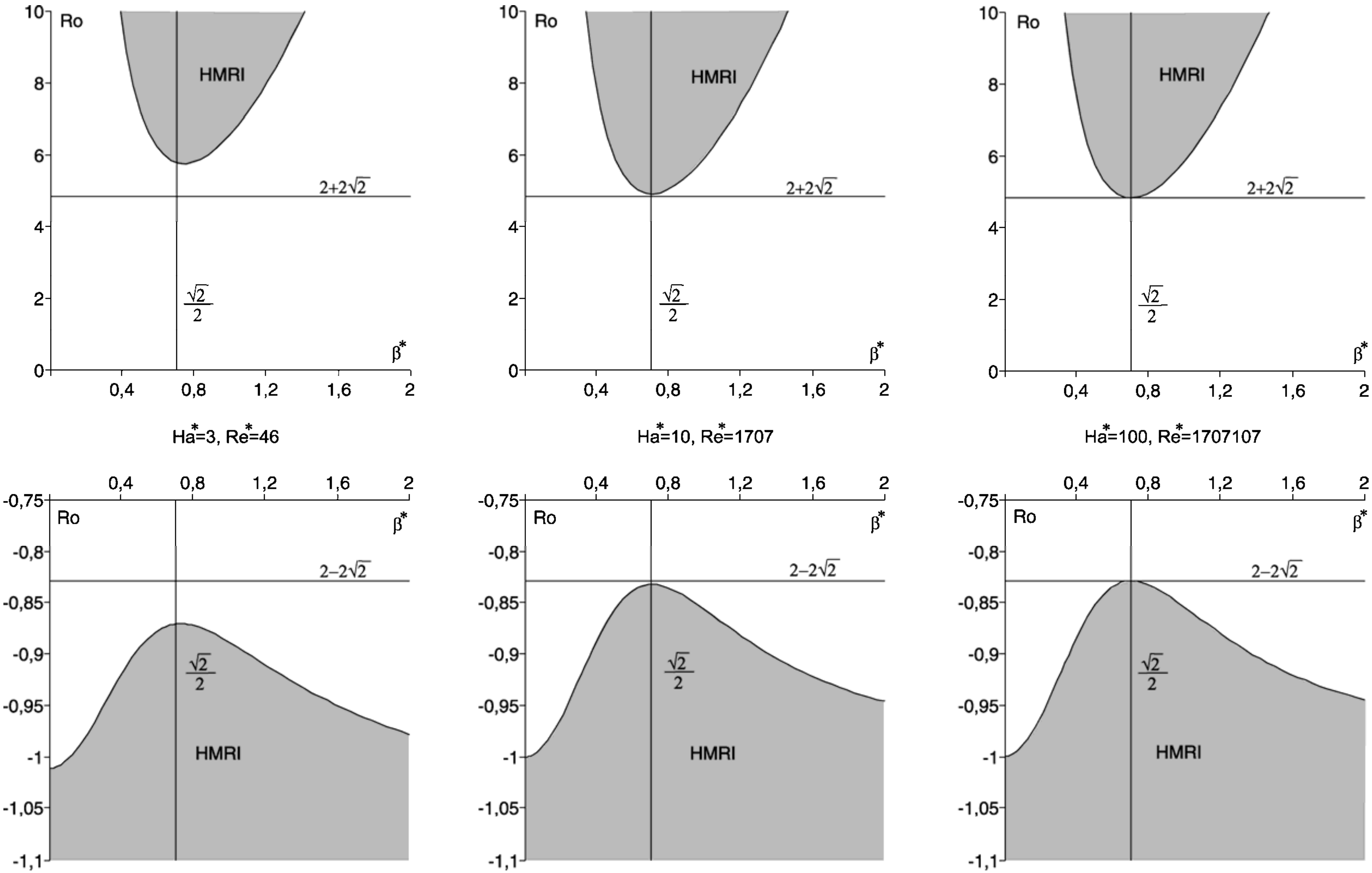
Induced magnetic field is negligible if

$$\mathrm{Pm} \ll \frac{1}{\mathrm{Re}} \quad \text{or, equivalently} \quad \mathrm{Rm} \ll 1$$

Only induced current matters

R. Hollerbach, G. Rudiger Phys. Rev. Lett. **95**(12), 124501 (2005)
J. Priede Phys. Rev. E **84**, 066314 (2011)

The upper and the lower Liu limits for inductionless HMRI at Rb=-1



O.N.K., F. Stefani The Astrophysical Journal 712, (2010) 52–68

Growth rates

Exact solutions in the inductionless limit, Pm=0

$$\lambda = -\frac{1}{\mathrm{Re}} - \mathrm{N}(1 - 2\beta^2\mathrm{Rb})$$

$$\pm 2\sqrt{-(1+\mathrm{Ro}) + i(\mathrm{Ro}+2)\beta\mathrm{N} + (\beta^2\mathrm{Rb}^2+1)\beta^2\mathrm{N}^2}$$

$\mathrm{N}=\mathrm{Ha}^2/\mathrm{Re}$ is the interaction parameter

Taylor series shows that HMRI is a perturbed inertial wave:

$$\lambda = \pm i2\sqrt{1+\mathrm{Ro}}$$

$$-\frac{1}{\mathrm{Re}} + \mathrm{N}\left(-1 + 2\beta^2\mathrm{Rb} \pm \frac{\beta(\mathrm{Ro}+2)}{\sqrt{1+\mathrm{Ro}}}\right) + O(\mathrm{N}^2)$$

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Inductionless approximation: $\mathrm{Pm} \to 0$

Stability: $\mathrm{Ro}^{-}(\mathrm{Rb},\beta) < \mathrm{Ro}(\mathrm{Ha},\mathrm{Re},\beta,\mathrm{Rb}) < \mathrm{Ro}^{+}(\mathrm{Rb},\beta)$

$\mathrm{Ro}^{\pm}(\mathrm{Rb},\beta)$ correspond to the limit $\mathrm{N} \to 0$

$$\mathrm{Ro}^{\pm} = -2 + \frac{1 - 2\beta^2\mathrm{Rb} \pm \sqrt{(1-2\beta^2\mathrm{Rb})^2 - 4\beta^2}}{2\beta^2(1-2\beta^2\mathrm{Rb})^{-1}}$$

Extrema: $\mathrm{Ro}_{cr}^{\pm}(\mathrm{Rb}) = -2 - 4\mathrm{Rb} \pm 2\sqrt{2\mathrm{Rb}(2\mathrm{Rb}+1)}$

Extremizers: $\beta_{cr} = \dfrac{-1}{\sqrt{-2\mathrm{Rb}}}, \quad \beta_{cr} = \dfrac{1}{\sqrt{-2\mathrm{Rb}}},$

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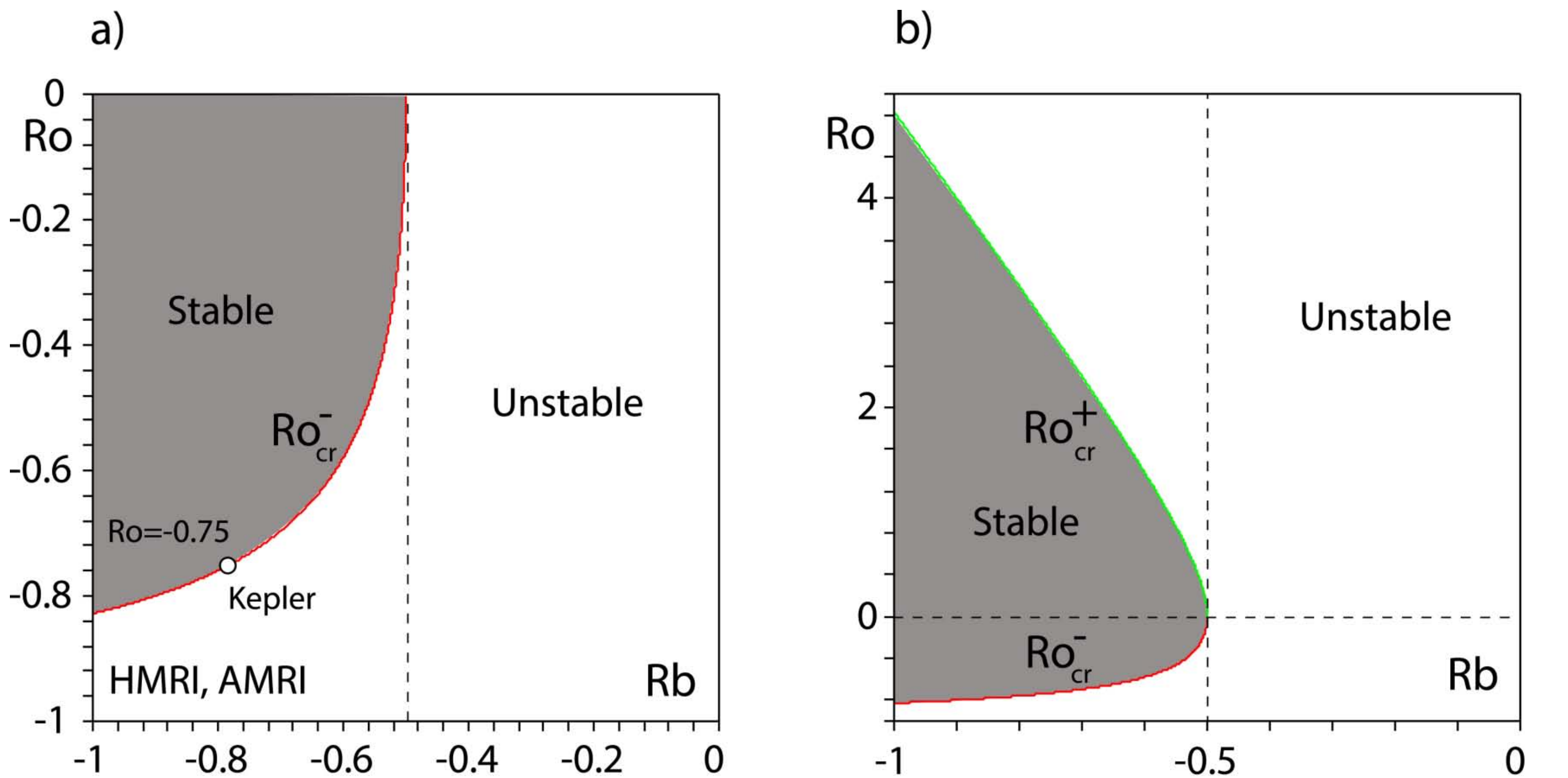
Extremal values for Ro in the inductionless limit



Liu limits (2006):

$$\mathrm{Rb} = -1, \quad \mathrm{Ro}^{\pm}_{cr}(-1) = 2 \pm 2\sqrt{2}$$

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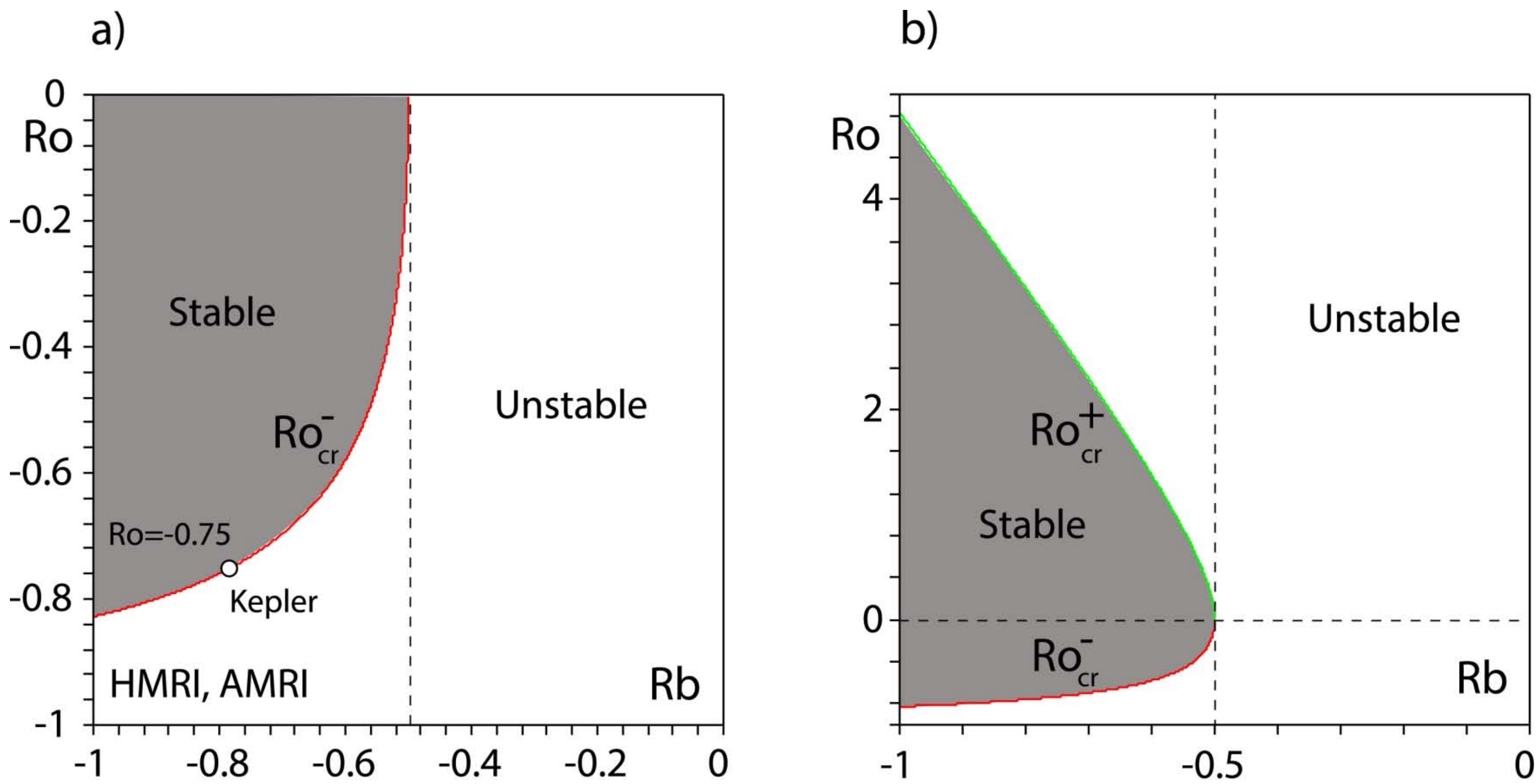
Extremal values for Ro in the inductionless limit



Generalization of the Liu limits to arbitrary Rb

$$\mathrm{Rb} = -\frac{1}{8}\frac{(\mathrm{Ro}+2)^2}{\mathrm{Ro}+1}$$

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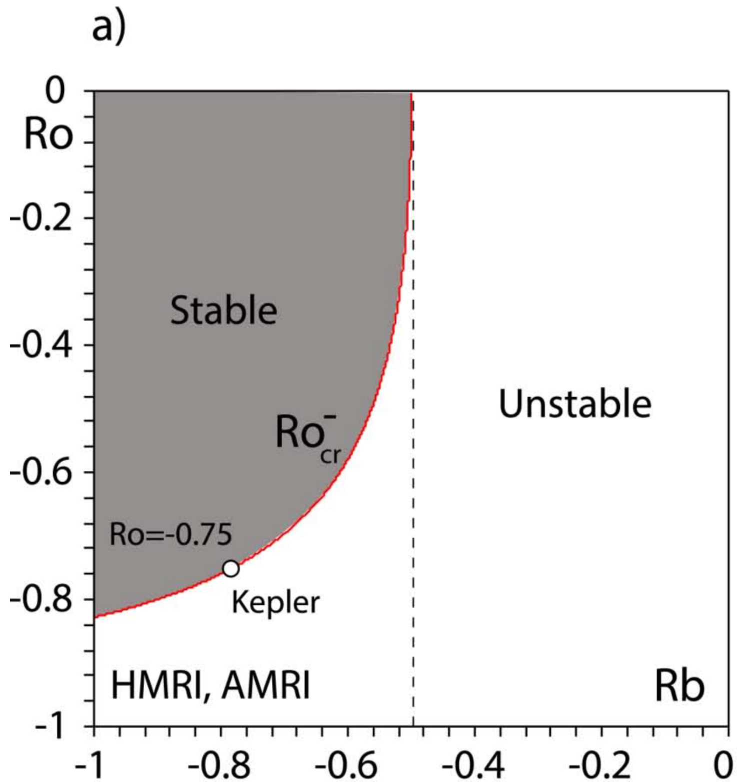
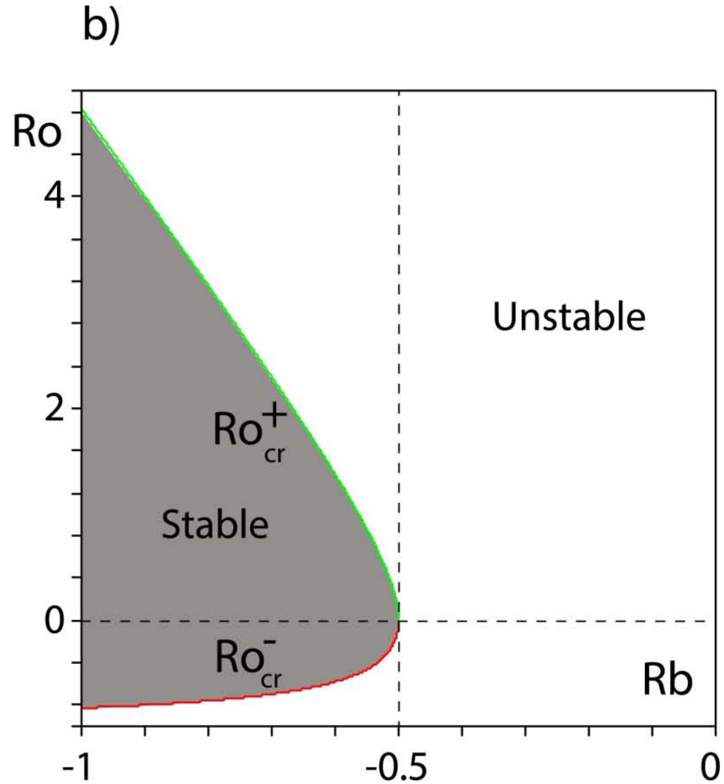
Extremal values for Ro in the inductionless limit



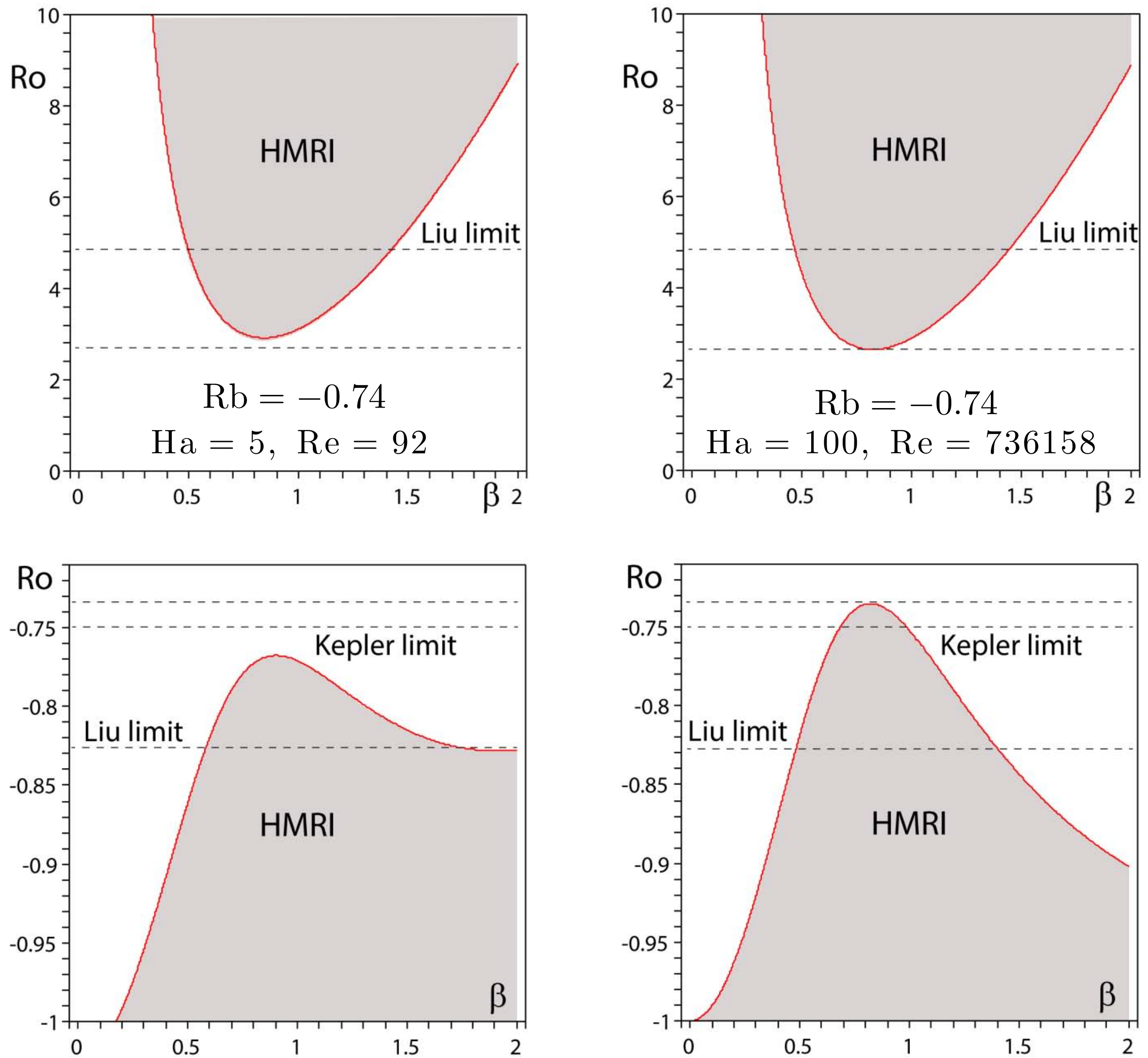


HMRI of a Keplerian flow

$$\mathrm{Rb} > -\frac{1}{8}\frac{(\mathrm{Ro}+2)^2}{\mathrm{Ro}+1}: \quad \mathrm{Ro} = -\frac{3}{4} \Rightarrow \mathrm{Rb} > -\frac{25}{32}$$

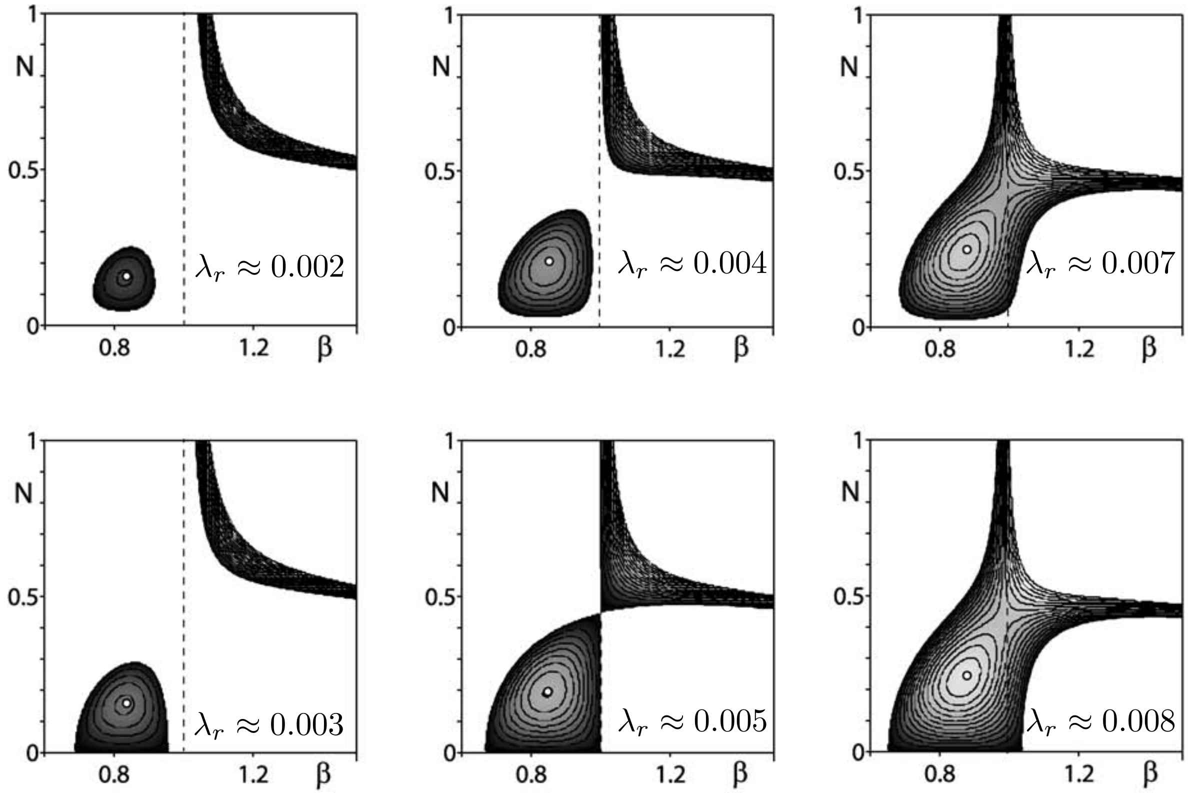
O.N.K., F. Stefani Phys. Rev. Lett. **111**(6), 061103 (2013)

Beyond the Liu limits



$\mathrm{Re} = 800$ a)

$\mathrm{Re} \to \infty$ b)



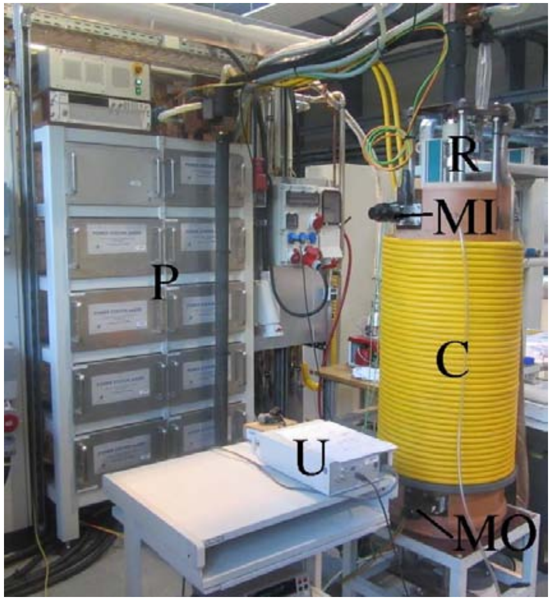
$\mathrm{Rb} = -0.76$ $\quad$ $\mathrm{Rb} = -0.75$ $\quad$ $\mathrm{Rb} = -0.74$

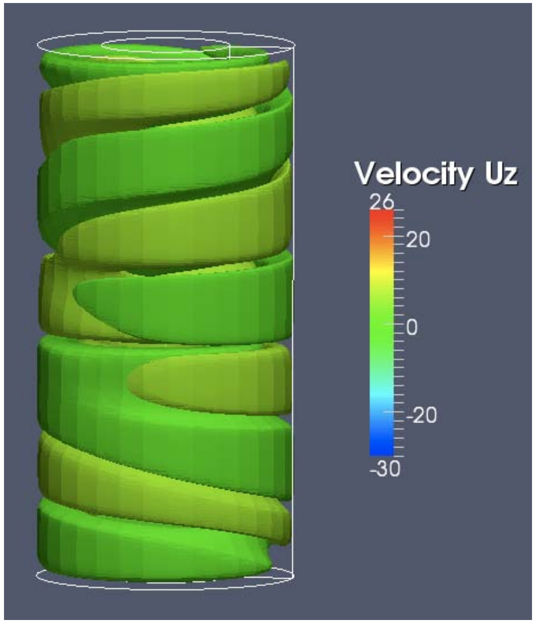
Growth rates of HMRI at Ro=-0.75 and Pm=0

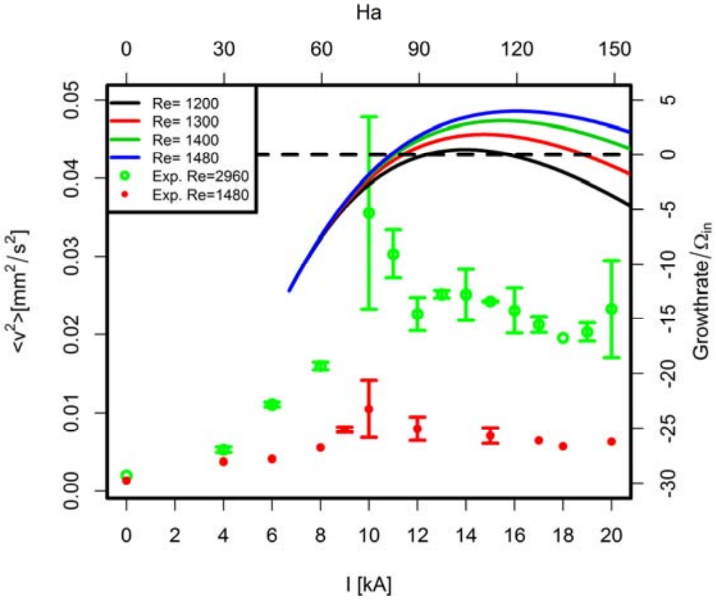
O.N.K., F. Stefani, Y. Fukumoto Journal of Fluid Mechanics, (2014) subm. arXiv:1401.8276

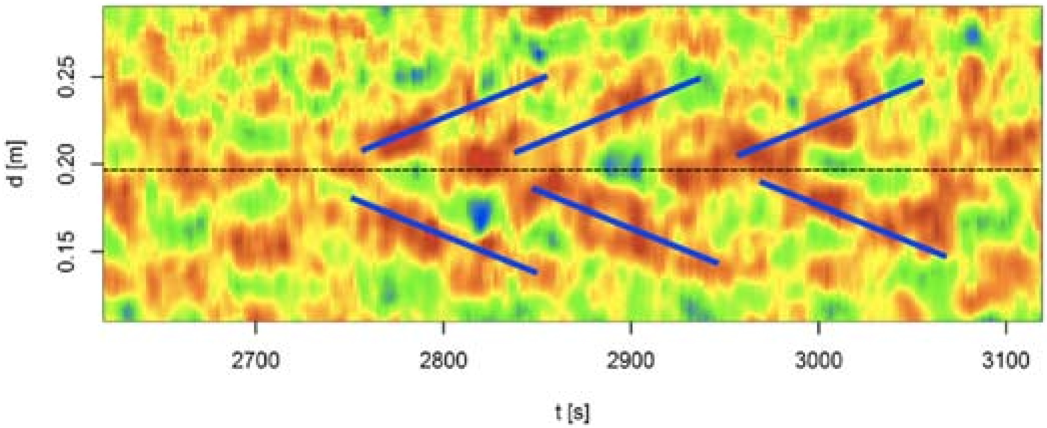
HZDR

First experimental evidence of azimuthal MRI with m=1,-1









Recently the PROMISE experiment has shown expected wave structure and confirmed the critical current of 10 kA

HZDR

Dispersion relation for non-axisymmetric perturbations

$$\det\left(\widetilde{H}-\lambda E\right)=0$$

λ is a complex growth rate measured in units of $\alpha\Omega$

$$\widetilde{H}=\begin{pmatrix} -in-\frac{1}{\mathrm{Re}} & 2\alpha & \frac{i\mathrm{Ha}(1+n\beta)}{\sqrt{\mathrm{ReRm}}} & \frac{-2\alpha\beta\mathrm{Ha}}{\sqrt{\mathrm{ReRm}}} \\ \frac{-2(1+\mathrm{Ro})}{\alpha} & -in-\frac{1}{\mathrm{Re}} & \frac{2\beta\mathrm{Ha}(1+\mathrm{Rb})}{\alpha\sqrt{\mathrm{ReRm}}} & \frac{i\mathrm{Ha}(1+n\beta)}{\sqrt{\mathrm{ReRm}}} \\ \frac{i\mathrm{Ha}(1+n\beta)}{\sqrt{\mathrm{ReRm}}} & 0 & -in-\frac{1}{\mathrm{Rm}} & 0 \\ \frac{-2\beta\mathrm{HaRb}}{\alpha\sqrt{\mathrm{ReRm}}} & \frac{i\mathrm{Ha}(1+n\beta)}{\sqrt{\mathrm{ReRm}}} & \frac{2\mathrm{Ro}}{\alpha} & -in-\frac{1}{\mathrm{Rm}} \end{pmatrix}$$

Parameters

azimuthal wave number

$$\beta=\alpha\frac{\omega_{A_\phi}}{\omega_A},\quad \mathrm{Re}=\alpha\frac{\Omega}{\omega_\nu},\quad \mathrm{Ha}=\frac{\omega_A}{\sqrt{\omega_\nu\omega_\eta}},\quad \boxed{n=\frac{m}{\alpha},}\quad \mathrm{Pm}=\frac{\omega_\nu}{\omega_\eta},\quad \mathrm{Rm}=\mathrm{RePm}$$

O.N. Kirillov, F. Stefani Phys. Rev. Lett. **111**(6), 061103 (2013)

HZDR

Growth rates of the azimuthal MRI in the inductionless limit (Pm=0)

New interaction parameter for the azimuthal field

$$\mathrm{N}_A := \beta^2 \mathrm{N}$$

Growth rates

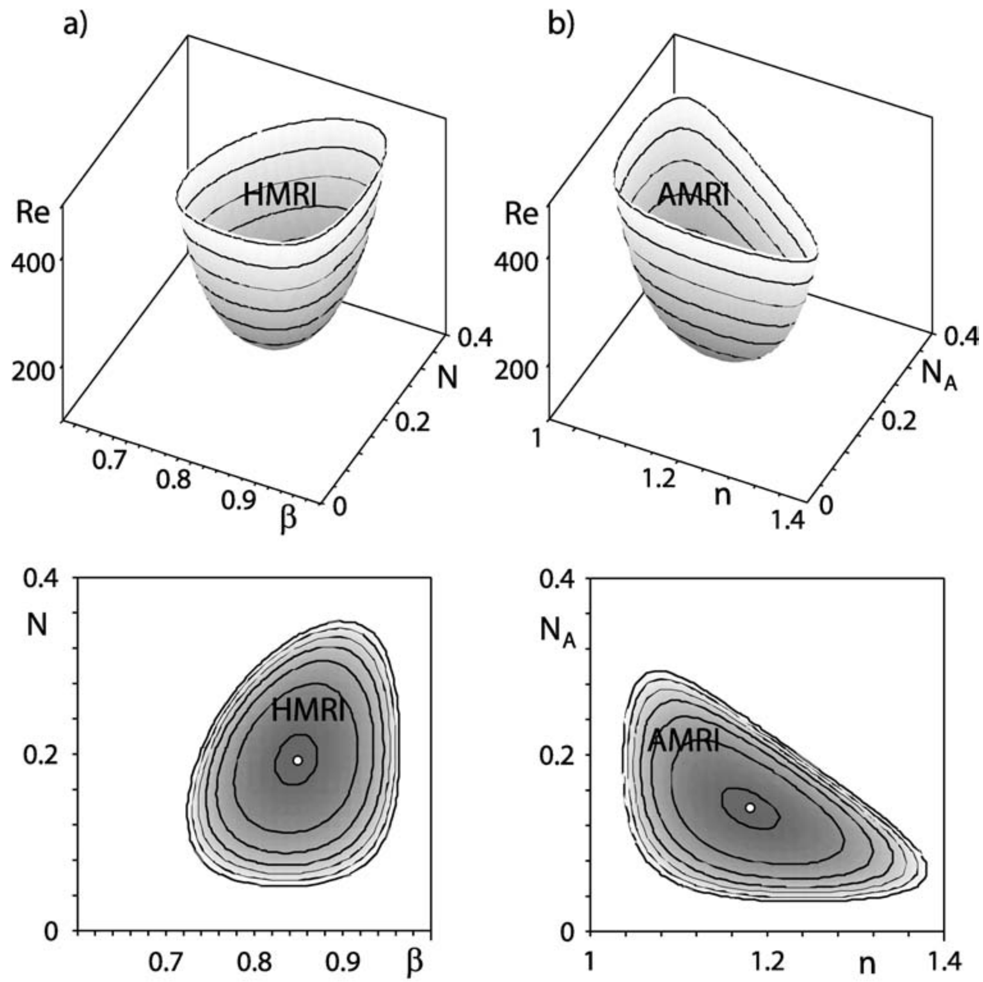
$$(\lambda_{1,2})_r = \mathrm{N}_A(2\mathrm{Rb} - n^2) - \frac{1}{\mathrm{Re}} \pm$$

$$\pm\sqrt{2}\sqrt{\mathrm{N}_A^2(\mathrm{Rb}^2 + n^2) - \mathrm{Ro} - 1 + \sqrt{\left(\mathrm{N}_A^2(\mathrm{Rb}^2 + n^2) - \mathrm{Ro} - 1\right)^2 + \mathrm{N}_A^2(\mathrm{Ro}+2)^2 n^2}}$$

O.N.K., F. Stefani, Y. Fukumoto Journal of Fluid Mechanics, (2014) subm. arXiv:1401.8276

Critical Reynolds numbers for HMRI and AMRI

at Ro=-0.75, Rb=-0.75



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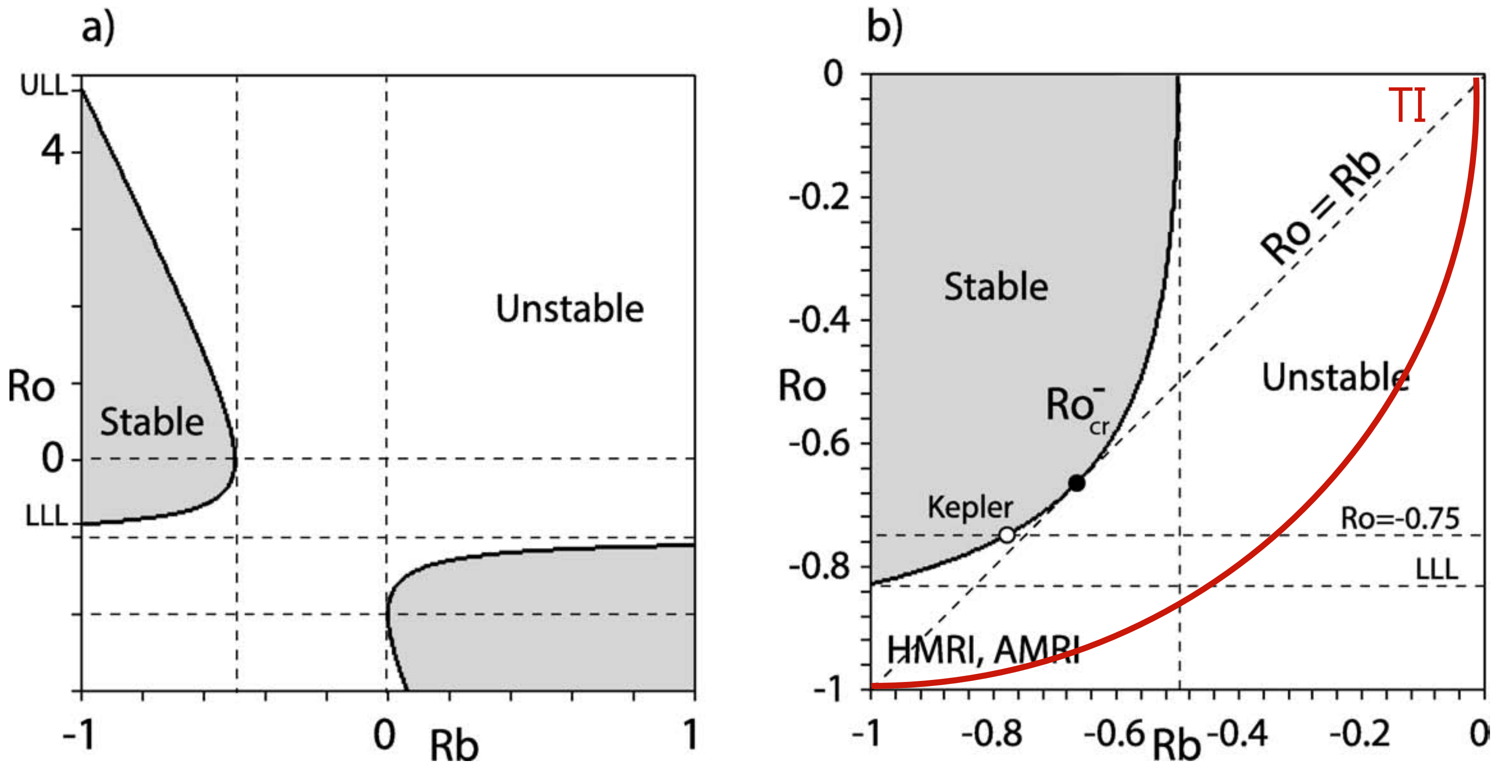
Transition from AMRI and the Tayler instability



The arc $\mathrm{Ro}=-\sqrt{-\mathrm{Rb}^2-2\mathrm{Rb}}$

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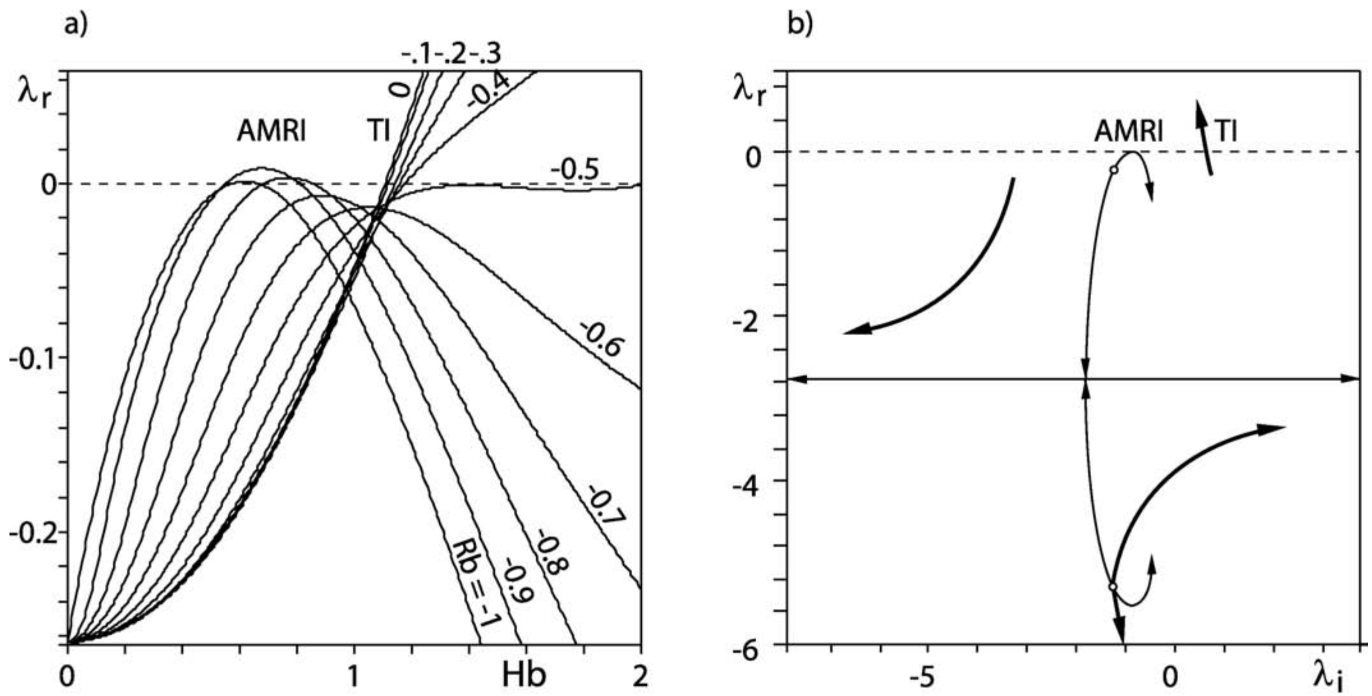
Growth rates of AMRI and TI along the arc



$$\mathrm{Hb} = \beta \mathrm{Ha}, \quad \mathrm{Pm} = 0.05, \quad n = 1.27842, \quad \mathrm{Re} = 3.8$$

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Instability threshold

$$\mathrm{Re}^2 = \frac{((1+\mathrm{Hb}^2 n^2)^2 - 4\mathrm{Hb}^2\mathrm{Rb}(1+\mathrm{Hb}^2 n^2) - 4\mathrm{Hb}^4 n^2)(1+\mathrm{Hb}^2(n^2-2\mathrm{Rb}))^2}{4(\mathrm{Hb}^4\mathrm{Ro}^2 n^2 - ((1+\mathrm{Hb}^2(n^2-2\mathrm{Rb}))^2 - 4\mathrm{Hb}^4 n^2)(\mathrm{Ro}+1))}.$$

The threshold of the Tayler in stability (Re=0, Rb=0)

$$\mathrm{Hb} = \frac{1}{\sqrt{1-(1-|n|)^2}}.$$

For the parameters in the TI experiment at Rossendorf

$$m=1, \quad k_z = \frac{2.4}{R}, \quad k_R = \frac{\pi}{R}, \quad \alpha \approx 0.6076, \quad \mathrm{Ha}_{exp} = \frac{|\boldsymbol{k}|^2 R^2}{\alpha}\mathrm{Hb} \approx 25.72\mathrm{Hb} \approx 30$$

This is close to numerical and experimental values

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Conclusions

- **Instability condition:** $$\mathrm{Rb} > -\frac{1}{8}\frac{(\mathrm{Ro}+2)^2}{\mathrm{Ro}+1}$$

$$\mathrm{Ro} = -\frac{3}{4}, \quad then \quad \mathrm{Rb} > -\frac{25}{32}$$

Inductionless MRI can destabilize Keplerian flows

- **New linear MHD instability for astrophysics**

- **Can be detected in liquid-metal experiments**

O.N. Kirillov, F. Stefani Phys. Rev. Lett. **111**(6), 061103 (2013)

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This work gives a complete overview on the subject of nonconservative stability from the modern point of view. Relevant mathematical concepts are presented, as well as rigorous stability results and numerous classical and contemporary examples from mechanics and physics.

It deals with both finite- and infinite-dimensional nonconservative systems and covers the fundamentals of the theory, including such topics as Lyapunov stability and linear stability analysis, Hamiltonian and gyroscopic systems, reversible and circulatory systems, influence of structure of forces on stability, and dissipation-induced instabilities, as well as concrete physical problems, including perturbative techniques for nonself-adjoint boundary eigenvalue problems, theory of the destabilization paradox due to small damping in continuous circulatory systems, Krein-space related perturbation theory for the MHD kinematic mean field α^2-dynamo, analysis of Campbell diagrams and friction-induced flutter in gyroscopic continua, non-Hermitian perturbation of Hermitian matrices with applications to optics, and magnetorotational instability and the Velikhov-Chandrasekhar paradox.

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