

Multiple eigenvalues in stability analysis

April 4, 2014

New York University, Courant Institute

Numerical Analysis and Scientific Computing Seminar

Oleg Kirillov

HzDR

 **HELMHOLTZ**
ZENTRUM DRESDEN
ROSSENDORF

Contents

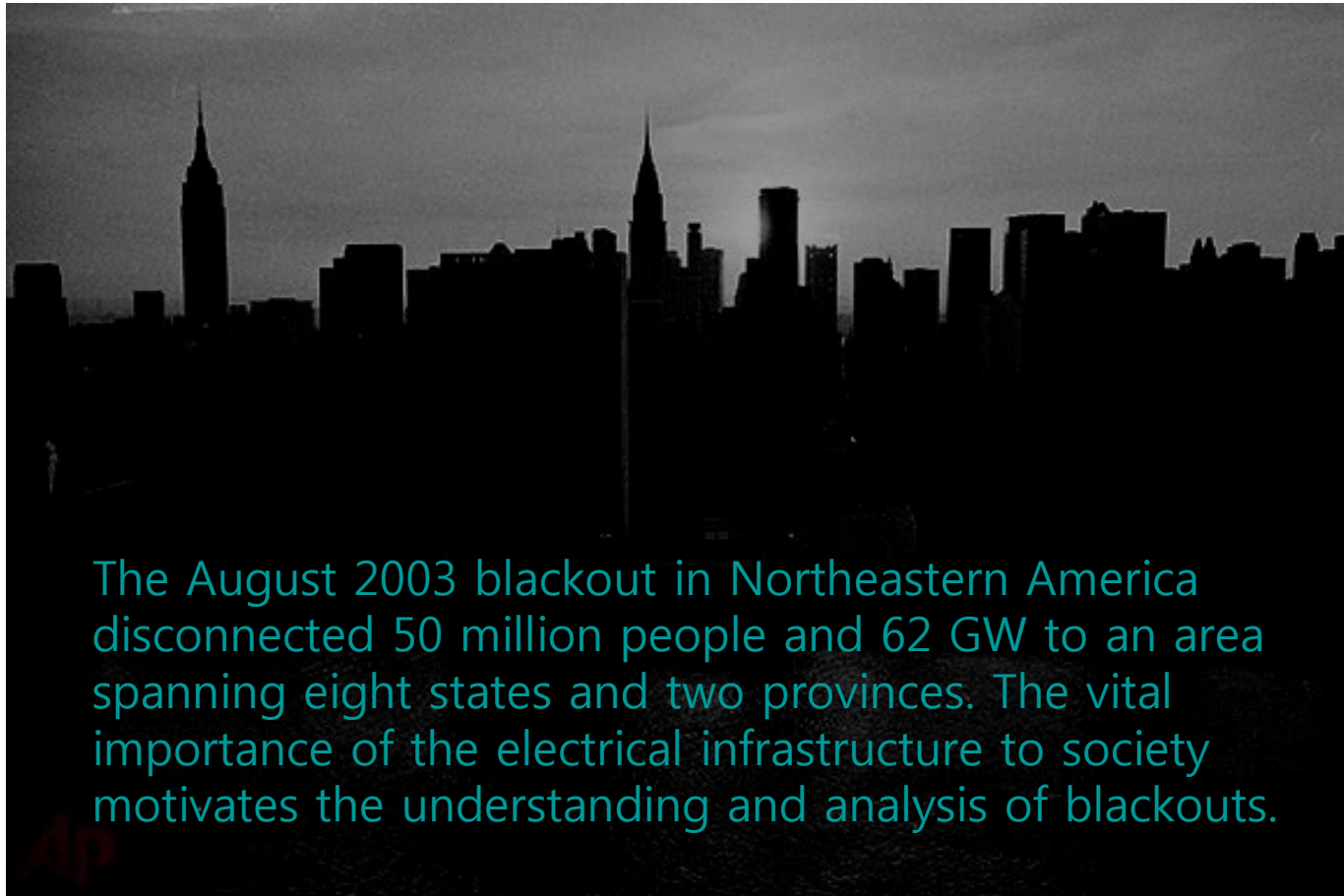
- Blackout in New York City
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- Friction-induced oscillations in an aircraft brake
- Weakly damped 1:1 resonance
- Dissipation-induced instabilities
- Ziegler-Bottema paradox and Whitney umbrella
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- Robust stability at the Swallowtail singularity
- Asymptotic stability and heavy damping

Blackout in NYC on August 15, 2003



James BARRON Power Surge Blacks Out Northeast [The New York Times](#), August 15, 2003 **HZDR**

Blackout in NYC on August 15, 2003



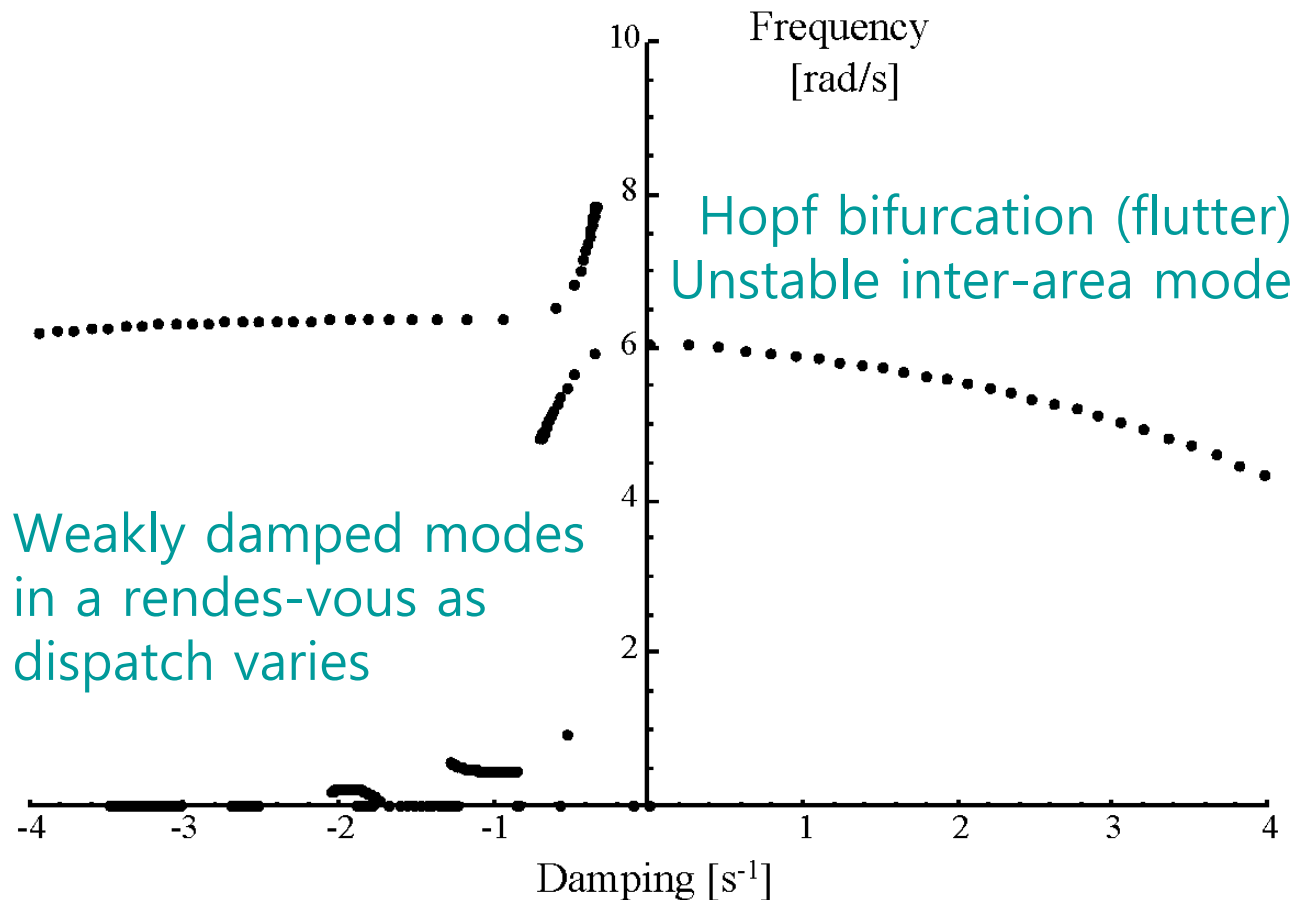
James BARRON Power Surge Blacks Out Northeast [The New York Times](#), August 15, 2003

Blackouts and the inter-area oscillations

- Fact: Power transactions are increasing in volume and variety.
Reason: Large amounts of money to be made in exploiting geographic differences in power prices.
Limitation: The onset of electro-mechanical inter-area oscillations.
- Disturbance: Synchronous (AC) generators rotors swing back and forth in search for a new operating point; while swinging, generators in one geographical area form a coherent group which oscillates against another coherent group in another geographical area.
The operation of the power system in the presence of a lightly damped inter-area mode is very difficult.
- Consequence: High power fluctuations on tie-lines that interconnect the areas. Even small variation of parameters may destabilize the oscillations and eventually lead to the power system collapse.
The inter-area oscillations have been involved in many blackouts.

Andrzej ADAMCZYK Damping of low frequency power system oscillations with wind power plants. Department of Energy Technology, Aalborg University, 2012, 239 pp.

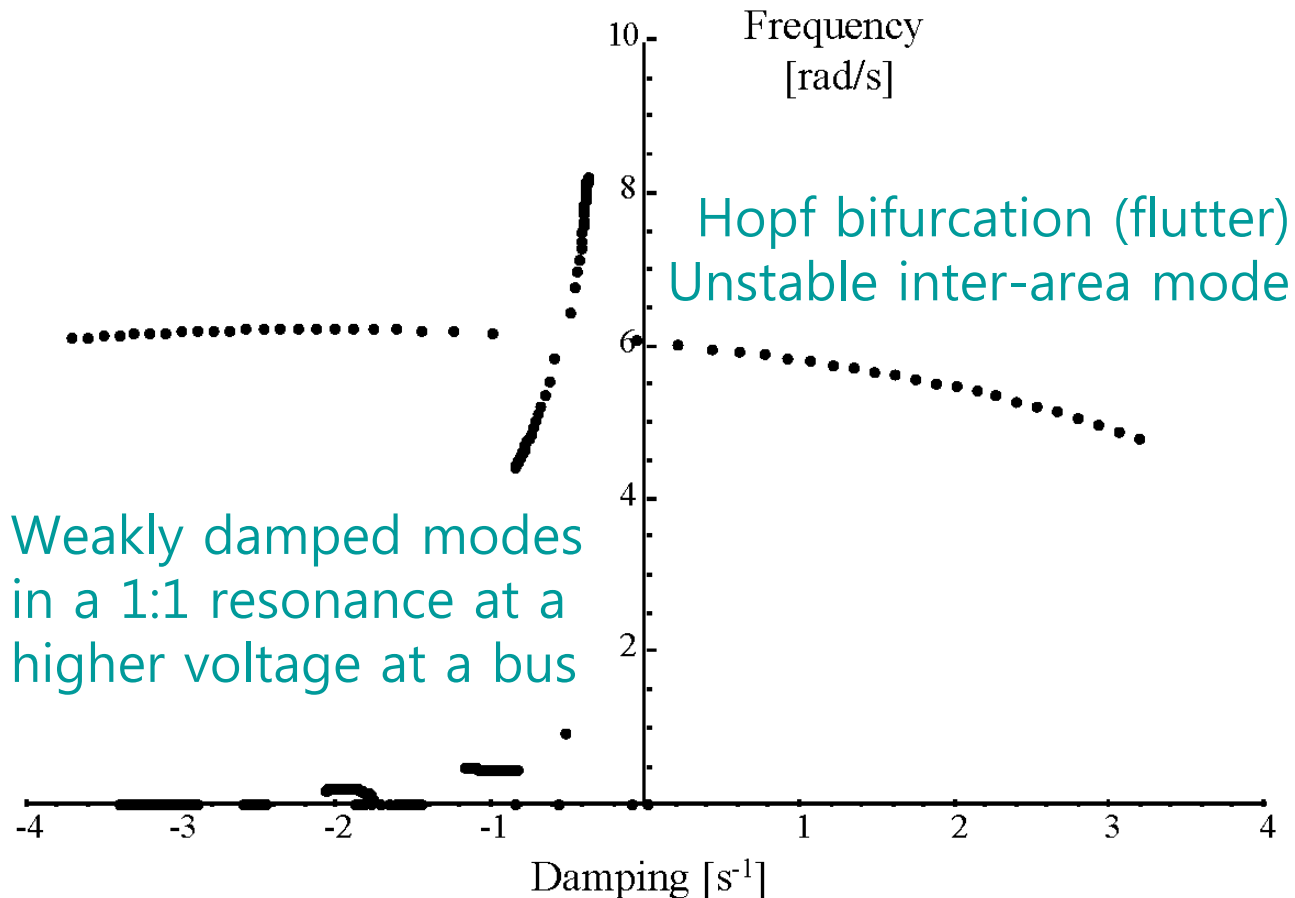
Modal resonance in a 3-bus power system



Ian DOBSON et al. Is strong modal resonance a precursor to power system oscillations?
[IEEE Trans. Circ. Syst. I 48 \(2001\) 340-349](#)

Aleksandar ZECEVIC, Dejan MILJKOVIC The effects of generation redispatch on Hopf
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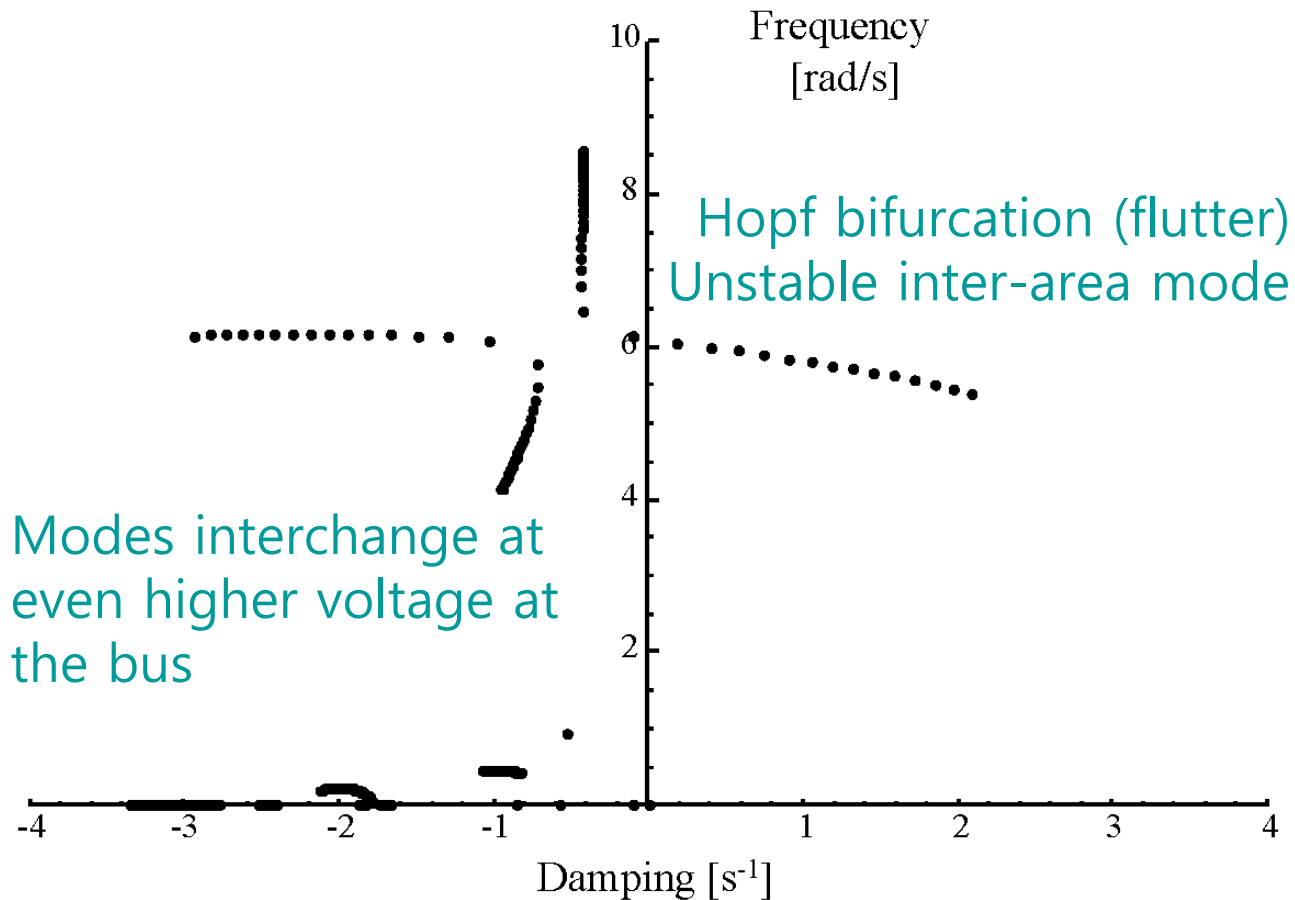
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Modal resonance in a 3-bus power system



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Interlude

„Unfortunately, it is quite common for an eigenvalue which is moving steadily towards a positive growth rate to suffer a sudden change of direction and subsequently fail to become unstable.

Similarly, it happens that modes which initially become more stable as a parameter increases change direction and subsequently achieve instability.

It is believed that these changes of direction are due to the nearby presence of multiple-eigenvalue points.”

Chris JONES Multiple eigenvalues and mode classification in plane Poiseuille flow.
Quart. J. Mech. Appl. Math. 41 (1988) 363-382

The classical power system model

$$M\ddot{\delta} + D\dot{\delta} + f_1(\delta, \phi, V, \mu) = 0, \quad f_2(\delta, \phi, V, \mu) = 0$$

M , symmetric matrix of generator rotor inertias;

D , symmetric damping matrix; δ , vector of generator internal bus angles; ϕ , vector of load bus angles; V , vector of constant power bus loads; μ , vector of network and load parameters

Non-symmetric matrix K of positional (circulatory) forces:

$$K^T \neq K := D_\delta f_1 - \begin{pmatrix} D_\phi \\ D_V \end{pmatrix} \begin{pmatrix} D_\phi \\ D_V \end{pmatrix}^{-1} D_\delta f_2$$

Linearized dynamics near the equilibrium $(\delta^*, \phi^*, V^*, \mu^*)$:

$$M\ddot{x} + D\dot{x} + Kx = 0, \quad x = \delta - \delta^*$$

This is a damped circulatory system

Harry KWATNY, Xiao-Ming YU Energy analysis of load-induced flutter instability in classical models of electric power networks. *IEEE Trans. Circ. Syst. I* 36 (1989) 1544-1557

Friction-induced whirl instability in an aircraft brake

Another example of a weakly damped circulatory system

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{F}_{\text{pressure}} + \mathbf{F}^{\text{NL}}(\mathbf{x}) \quad (20)$$

where $\ddot{\mathbf{x}}$, $\dot{\mathbf{x}}$ and $\mathbf{x}$ are the acceleration, velocity, and displacement response 15-dimensional vectors of the degrees-of-freedom, respectively. $\mathbf{M}$ is the mass matrix, $\mathbf{C}$ is

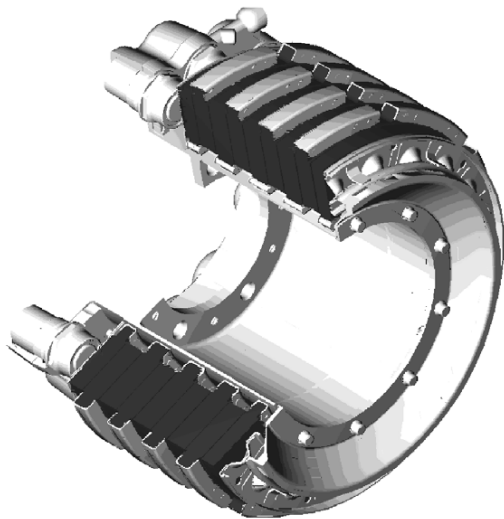


Fig. 2. Aircraft brake system.

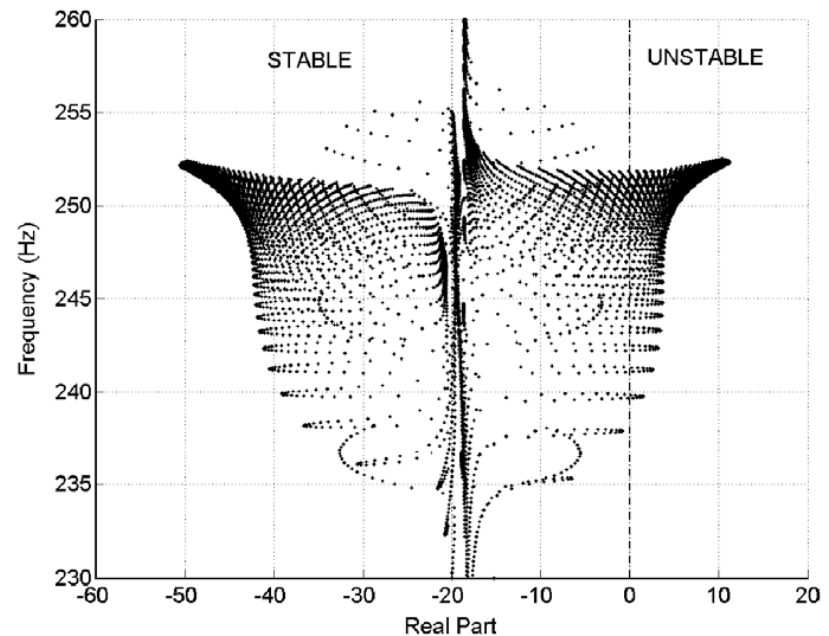


Fig. 11. Evolution of the imaginary and real part for the stable/unstable mode versus the pressure and the friction coefficient in the complex plane.

J.-J. SINOUE et al. Stability analysis and non-linear behaviour of structural systems using the complex non-linear modal analysis *Computers and Structures* 84 (2006) 1891–1905

Sophisticated effects of weak damping

1878 Kelvin

stability of rotating ellipsoidal shells containing fluid



1880 Greenhill

prolate shells unstable, oblate stable



1942 Sobolev

instability of chemical artillery shells



1944 Pontryagin, 1950 Krein

Hilbert space with indefinite metric,
Krein collision, Krein signature



1960 Sturrock

dissipation-induced instability of
negative energy modes in
Hamiltonian systems

1883 Greenhill

buckling of a screw-shaft of a steamer



1927 Nicolai

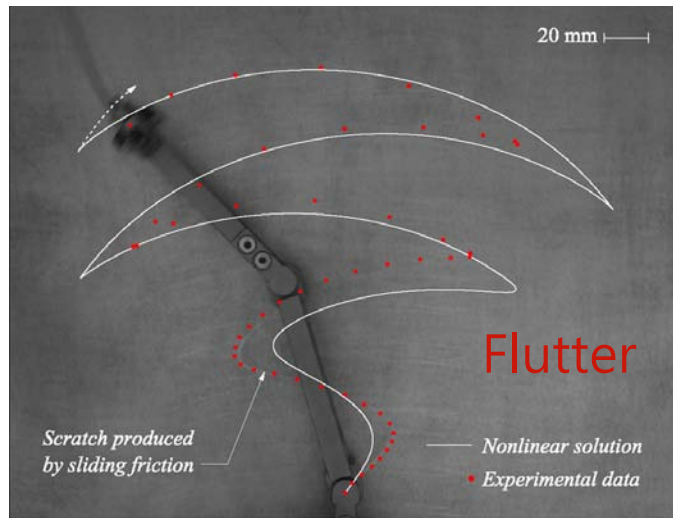
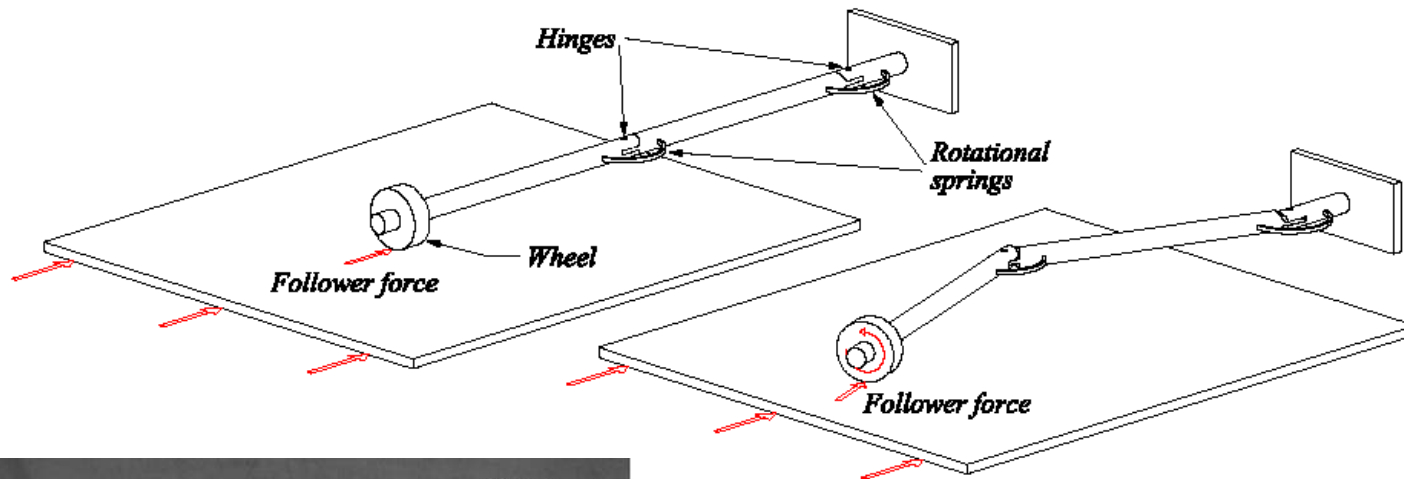
buckling and flutter of shafts under
compression and (also follower) torque



1952 Ziegler

flutter of rods under follower force,
**destabilization paradox due to small
dissipation in circulatory systems**

Ziegler's pendulum: Experimental realization

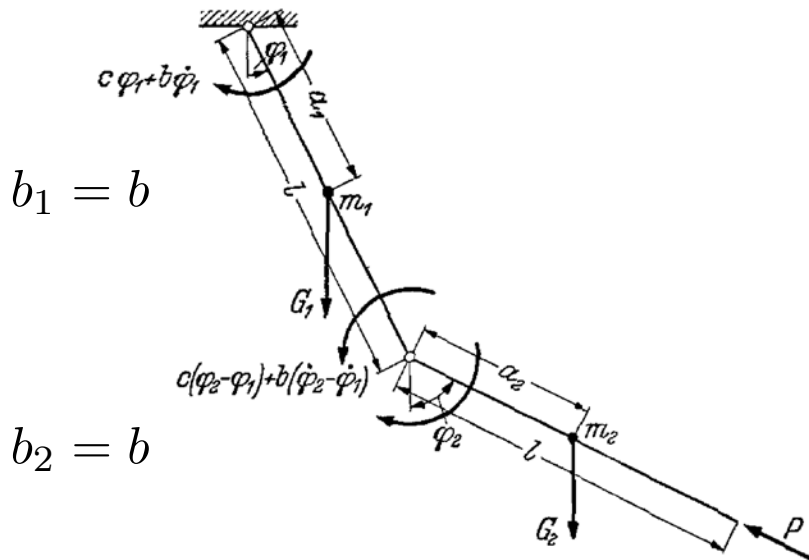


The non-conservative follower force originates through a wheel of negligible mass mounted at the top of the double pendulum and constrained to slide against a moving frictional plane

Davide BIGONI, Giovanni NOSELLI, Experimental evidence of flutter and divergence instabilities induced by dry friction, *Journal of the Mechanics and Physics of Solids*, 59 (2011) 2208–2226

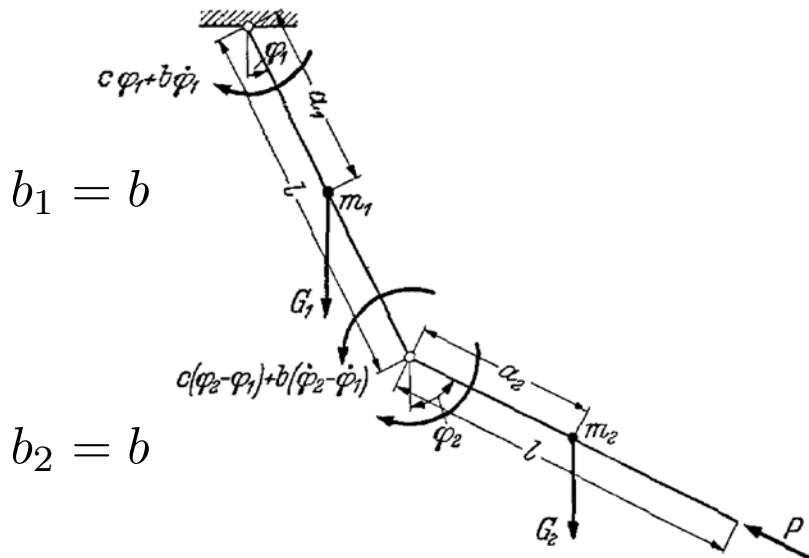
Destabilization paradox in circulatory systems

- Ziegler's pendulum



Destabilization paradox in circulatory systems

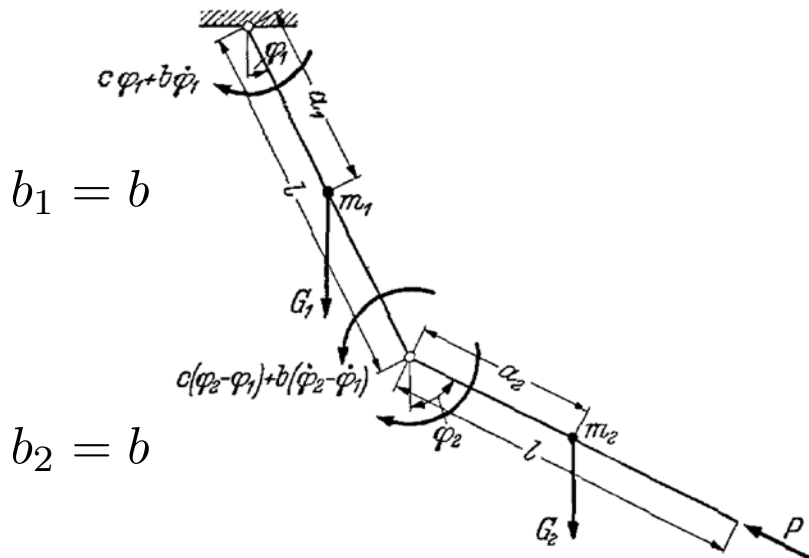
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- Follower force, P

Destabilization paradox in circulatory systems

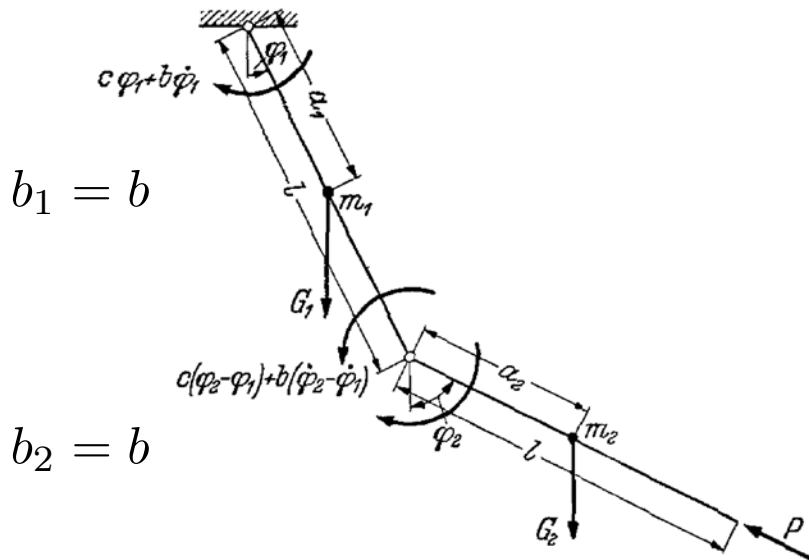
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- Follower force, P
- Stiffness, c

Destabilization paradox in circulatory systems

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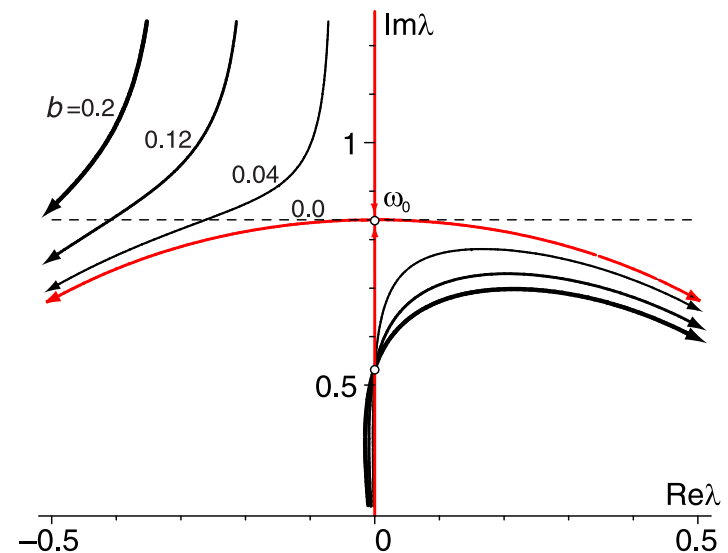
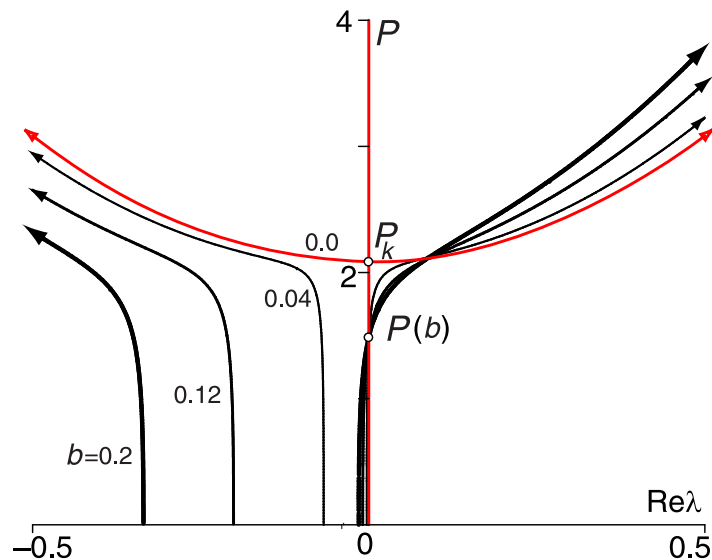


- Follower force, P
- Stiffness, c
- Damping, b

Hans ZIEGLER Die Stabilitätskriterien der Elastomechanik Ing.-Arch. 20 (1952) 49-56

Destabilization paradox in circulatory systems

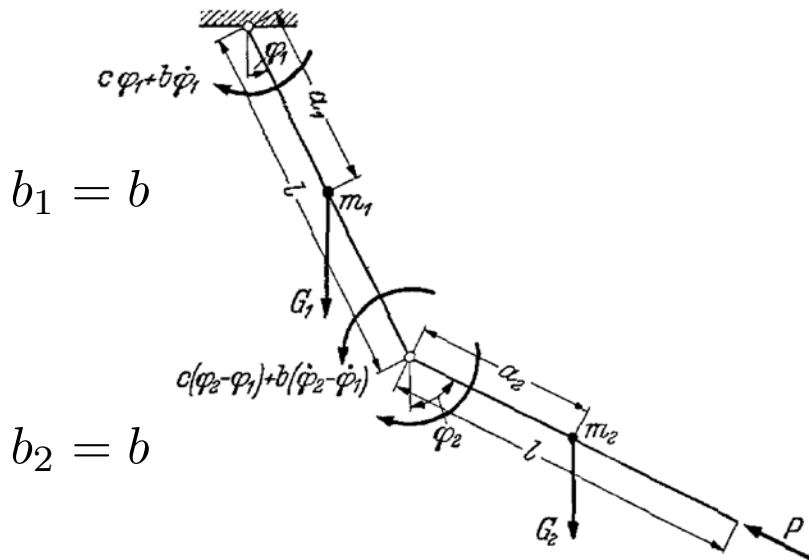
Imperfect modal resonance in the presence of damping



Imaginary eigenvalues get **positive real** increments under a dissipative perturbation

Destabilization paradox in circulatory systems

- Ziegler's pendulum



- Follower force, P
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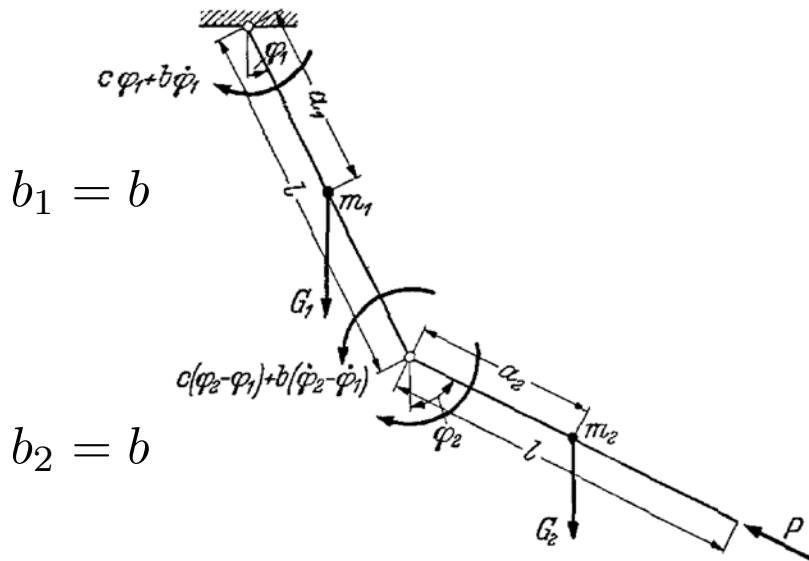
- Stability of the vertical equilibrium



Hans ZIEGLER Die Stabilitätskriterien der Elastomechanik Ing.-Arch. 20 (1952) 49-56

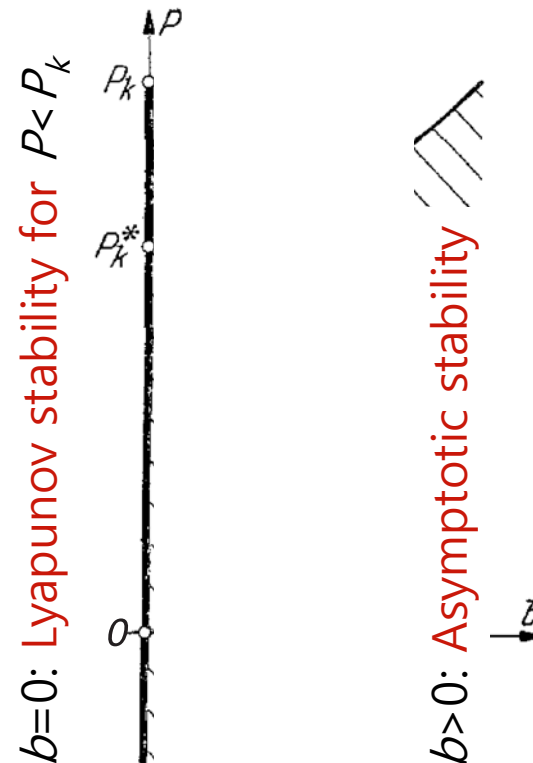
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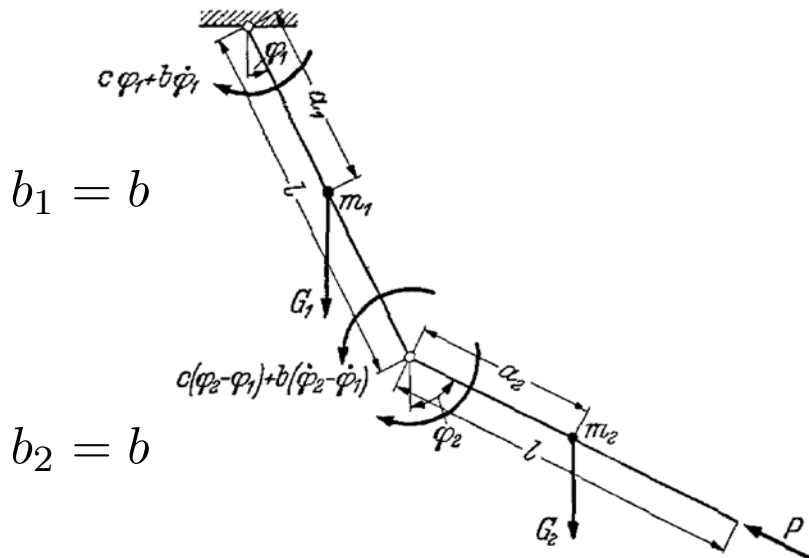
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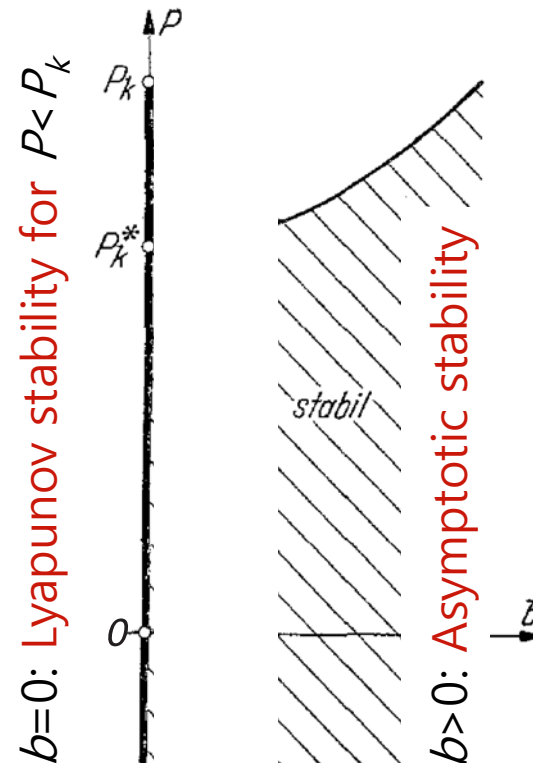
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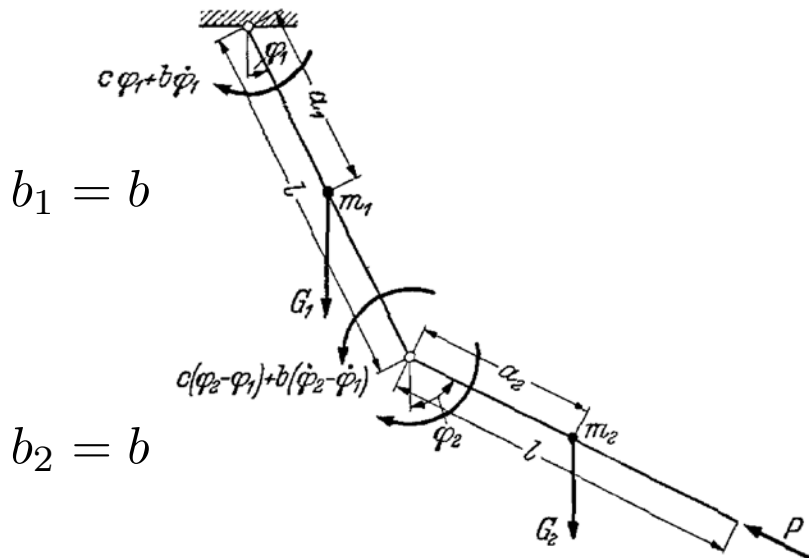


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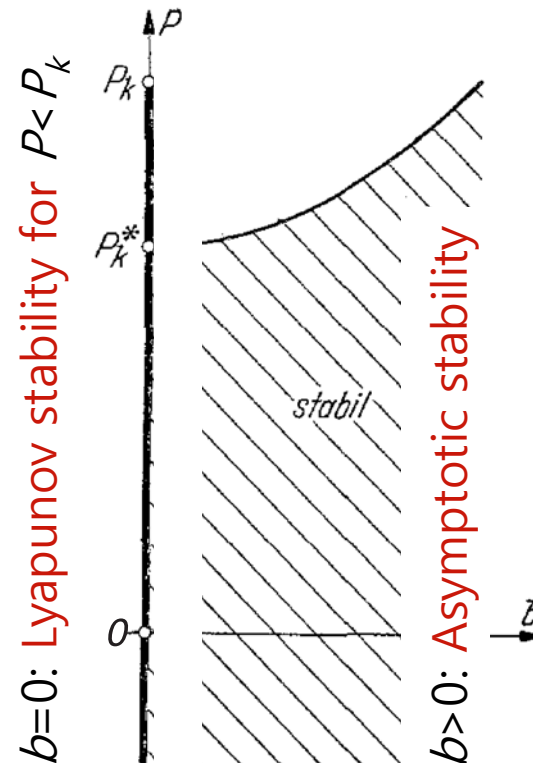
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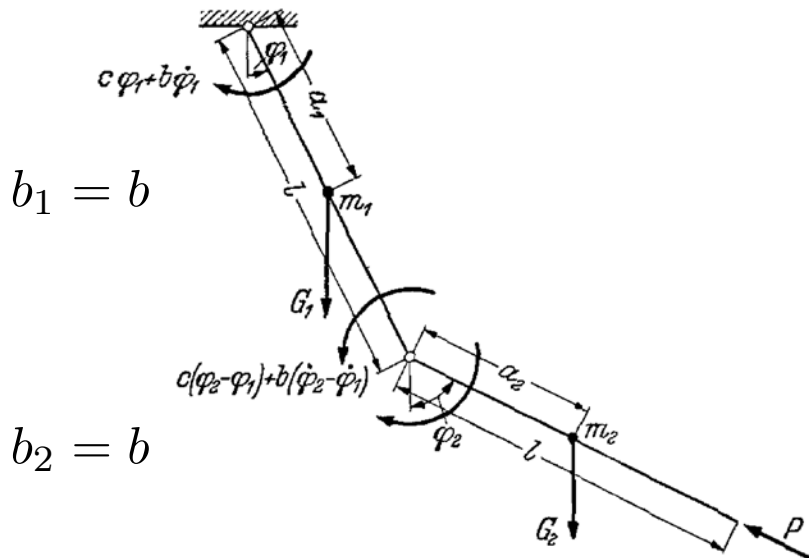
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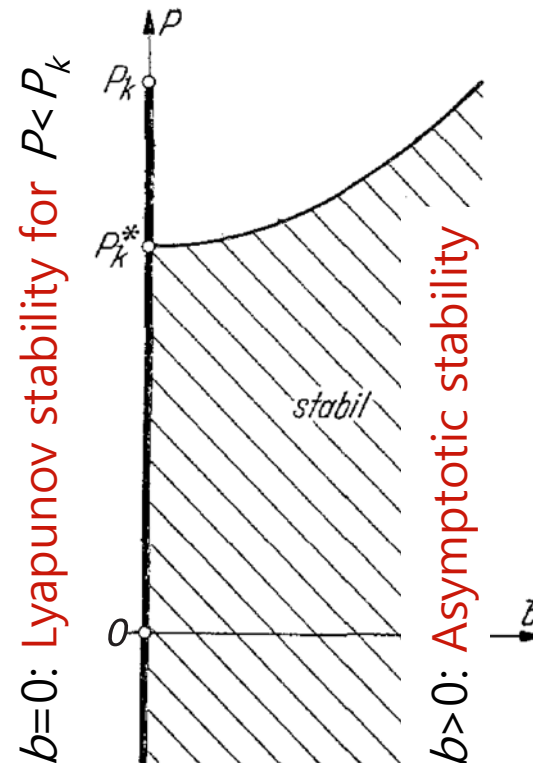
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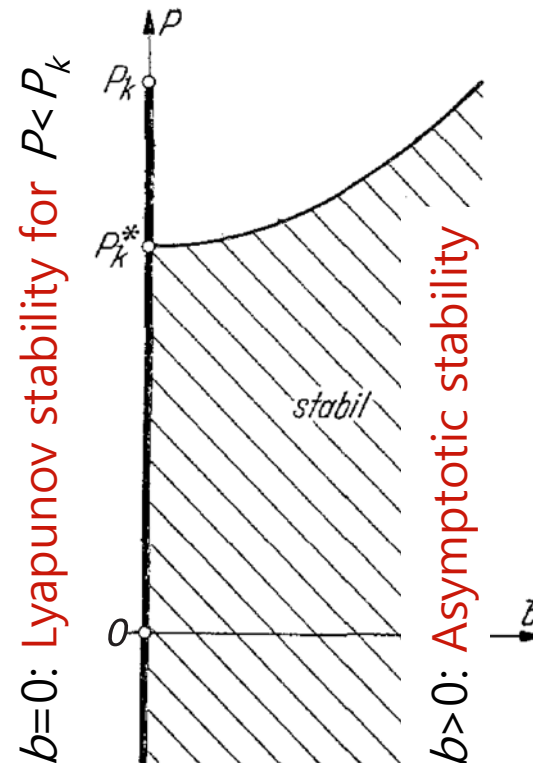
Destabilization paradox in circulatory systems

- Ziegler's pendulum

$$m_1 = 2m, \quad m_2 = m$$

$$a_1 = a_2 = l, \quad b_1 = b_2 = b$$

- Stability of the vertical equilibrium



Hans ZIEGLER Die Stabilitätskriterien der Elastomechanik Ing.-Arch. 20 (1952) 49-56

Destabilization paradox in circulatory systems

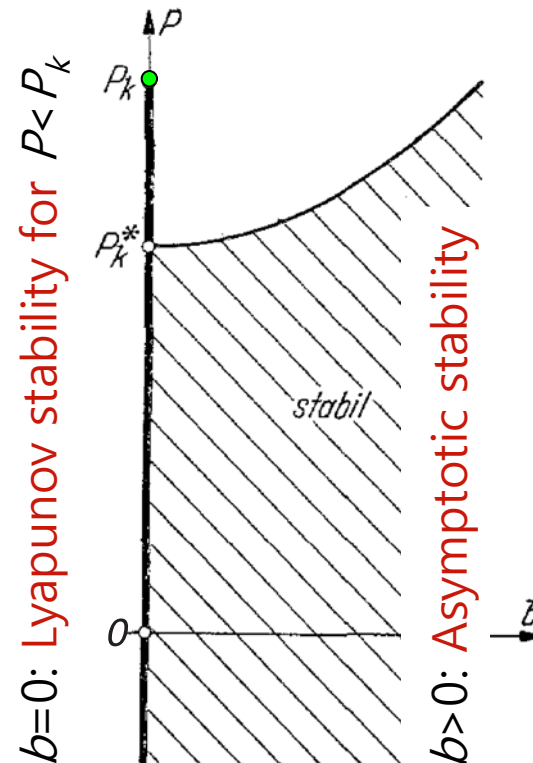
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$$m_1 = 2m, \quad m_2 = m$$

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$$b = 0 : \quad P_k = \left(\frac{7}{2} - \sqrt{2} \right) \frac{c}{l} \approx 2.086 \frac{c}{l}$$

- Stability of the vertical equilibrium



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Destabilization paradox in circulatory systems

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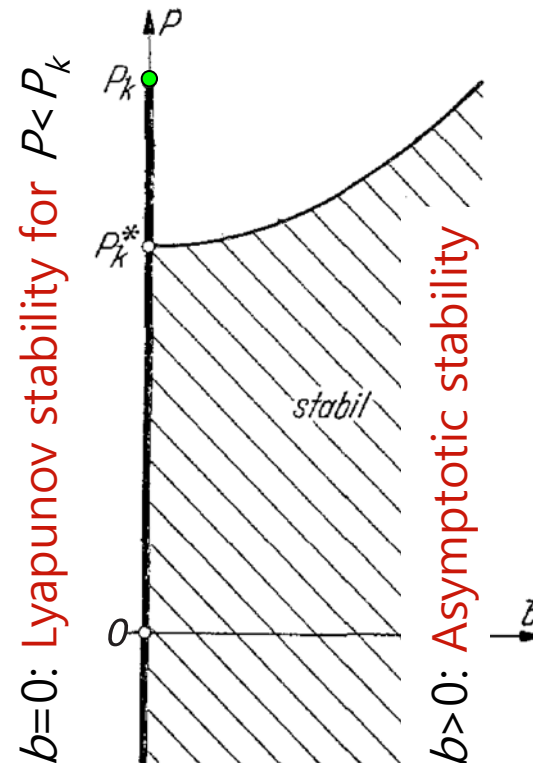
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$$b > 0 : \quad P(b) = \frac{41}{28} \frac{c}{l} + \frac{1}{2} \frac{b^2}{ml^3}$$

- Stability of the vertical equilibrium



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Destabilization paradox in circulatory systems

■ Ziegler's pendulum

$$m_1 = 2m, \quad m_2 = m$$

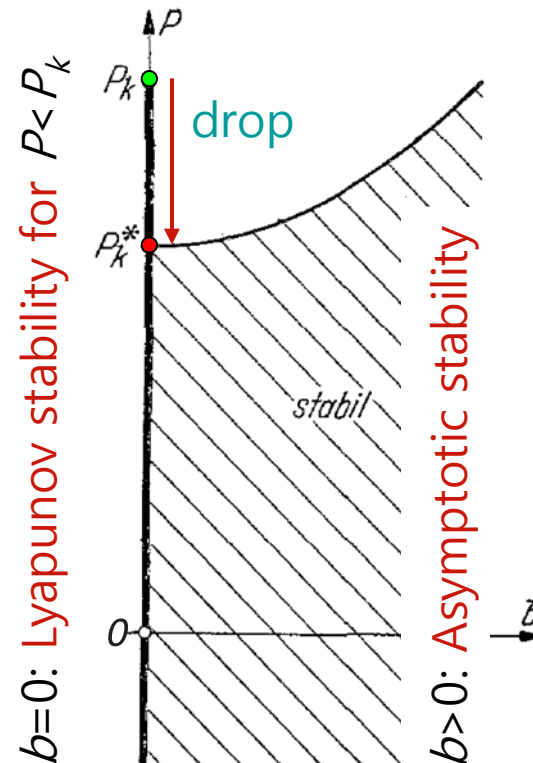
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$$P_k^* = \lim_{b \rightarrow 0} P(b) = \frac{41}{28} \frac{c}{l} \approx 1.464 \frac{c}{l} < P_k$$

■ Stability of the vertical equilibrium



Hans ZIEGLER Die Stabilitätskriterien der Elastomechanik Ing.-Arch. 20 (1952) 49-56

Universality of the effects of weak dissipation

- "Will the limit of stability corresponding to a gradually vanishing damping coincide in the limit with that found on the assumption that there is no damping?"

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- **The greatest theoretical interest** is evidently centered in the unique effect of damping in the presence of non-potential forces, and in particular, in the differences in the results for systems with slight damping which then becomes zero and systems in which damping is absent from the start.

Universality of the effects of weak dissipation

- "Will the limit of stability corresponding to a gradually vanishing damping coincide in the limit with that found on the assumption that there is no damping?"
- In the case of conservative forces the answer is that it will.
- **The greatest theoretical interest** is evidently centered in the unique effect of damping in the presence of non-potential forces, and in particular, in the differences in the results for systems with slight damping which then becomes zero and systems in which damping is absent from the start.
- These interesting aspects require further study for obtaining further, more definite, results."

Pioneering works by Bottema (1956)

A linear non-conservative system with 2 d.o.f.

$$\ddot{\mathbf{x}} + (\mathbf{D} + \mathbf{G})\dot{\mathbf{x}} + (\mathbf{K} + \mathbf{N})\mathbf{x} = 0$$

Forces:

Dissipative, $\mathbf{D} = \begin{pmatrix} d_{11} & d_{12} \\ d_{12} & d_{22} \end{pmatrix}$ Potential, $\mathbf{K} = \begin{pmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{pmatrix}$

Gyroscopic, $\mathbf{G} = \begin{pmatrix} 0 & \Omega \\ -\Omega & 0 \end{pmatrix}$ Circulatory, $\mathbf{N} = \begin{pmatrix} 0 & \nu \\ -\nu & 0 \end{pmatrix}$

Oene BOTTEMA The Routh-Hurwitz condition for the biquadratic equation *Indagationes Mathematicae*, 18 (1956) 403-406

Pioneering works by Bottema (1956)

A linear non-conservative system with 2 d.o.f.

$$\ddot{\mathbf{x}} + (\mathbf{D} + \mathbf{G})\dot{\mathbf{x}} + (\mathbf{K} + \mathbf{N})\mathbf{x} = 0$$

Characteristic polynomial:

$$\mathbf{x} = e^{\mu t} \mathbf{u}, \quad q(\mu) = \mu^4 + q_1 \mu^3 + q_2 \mu^2 + q_3 \mu + q_4$$

$$q_1 = \text{tr} \mathbf{D}, \quad q_3 = \text{tr} \mathbf{K} \text{tr} \mathbf{D} - \text{tr} \mathbf{K} \mathbf{D} + 2\Omega \nu$$

$$q_2 = \text{tr} \mathbf{K} + \det \mathbf{D} + \Omega^2, \quad q_4 = \det \mathbf{K} + \nu^2$$

Oene BOTTEMA The Routh-Hurwitz condition for the biquadratic equation *Indagationes Mathematicae*, 18 (1956) 403-406

Pioneering works by Bottema (1956)

$$q(\mu) = \mu^4 + q_1\mu^3 + q_2\mu^2 + q_3\mu + q_4$$

Hurwitz condition:

$$q_i > 0, \quad q_2 > \frac{q_1^2 q_4 + q_3^2}{q_1 q_3}$$

Oene BOTTEMA The Routh-Hurwitz condition for the biquadratic equation *Indagationes Mathematicae*, 18 (1956) 403-406

Pioneering works by Bottema (1956)

$$q(\mu) = \mu^4 + q_1\mu^3 + q_2\mu^2 + q_3\mu + q_4$$

$$\mu = c\lambda, \quad c = \sqrt[4]{q_4}, \quad a_i = \frac{q_i}{c^i}$$

Hurwitz condition:

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Pioneering works by Bottema (1956)

$$p(\lambda) = \lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + 1$$

$$\mu = c\lambda, \quad c = \sqrt[4]{q_4}, \quad a_i = \frac{q_i}{c^i}$$

Hurwitz condition:

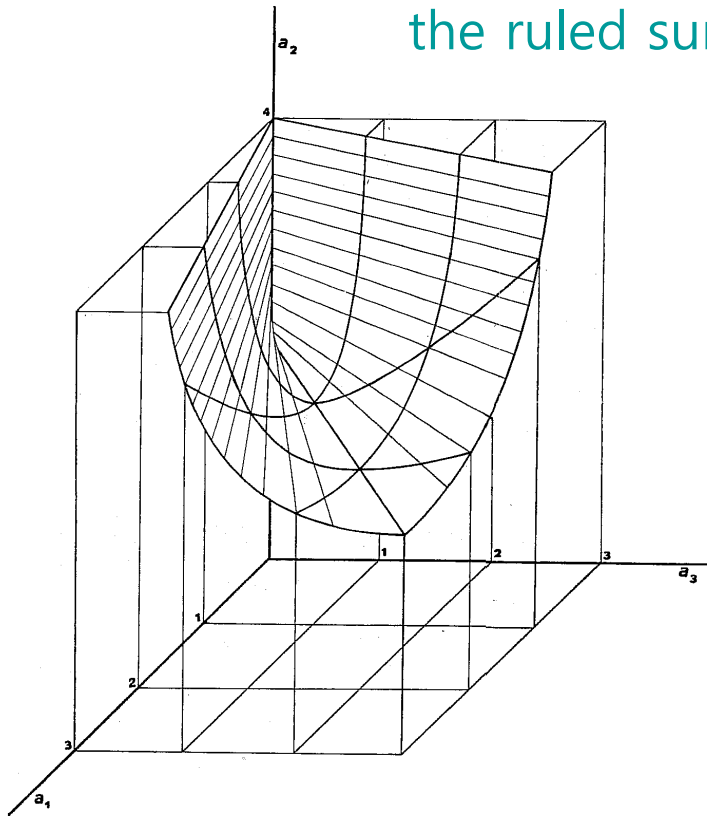
$$a_i > 0, \quad a_2 > 2 + \frac{(a_1 - a_3)^2}{a_1 a_3}$$

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Asymptotic stability inside
the ruled surface



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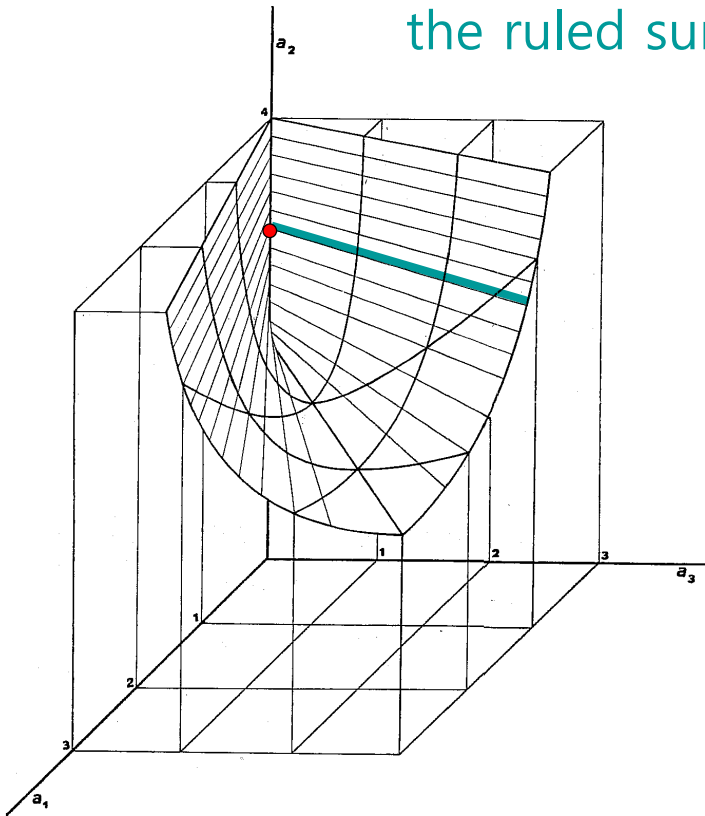
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Generators:

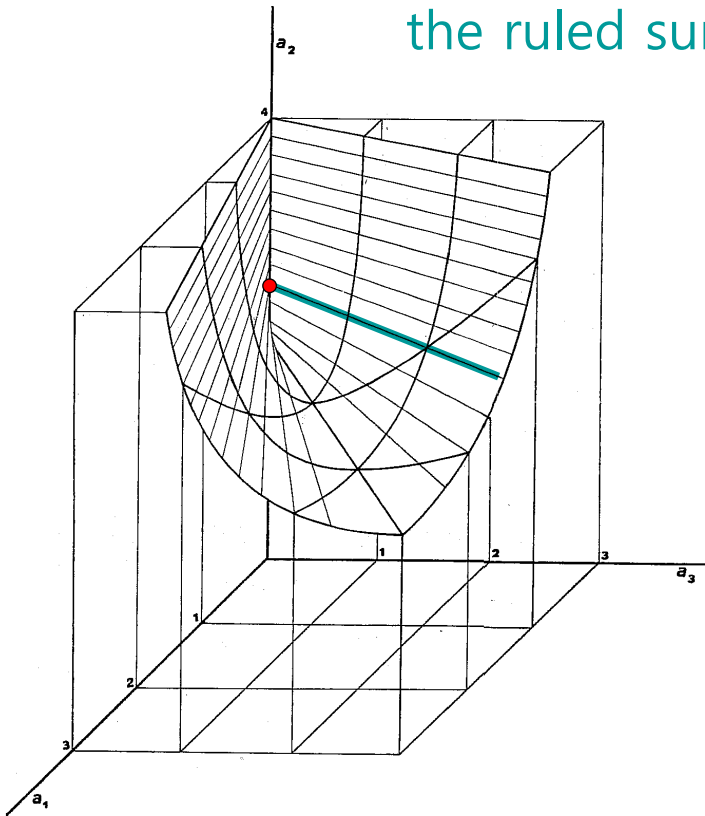
$$a_3 = r a_1, \quad a_2 = r + \frac{1}{r}$$

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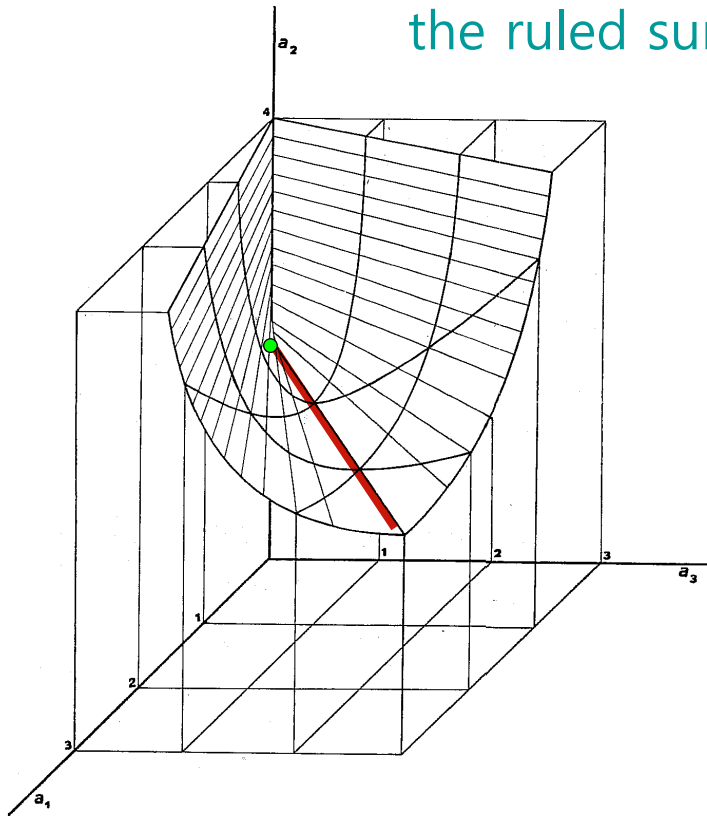
$$r \in (0, \infty)$$

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Generators:

$$a_3 = r a_1, \quad a_2 = r + \frac{1}{r}$$

$$r \in (0, \infty)$$

Minimum: $a_2 = 2$

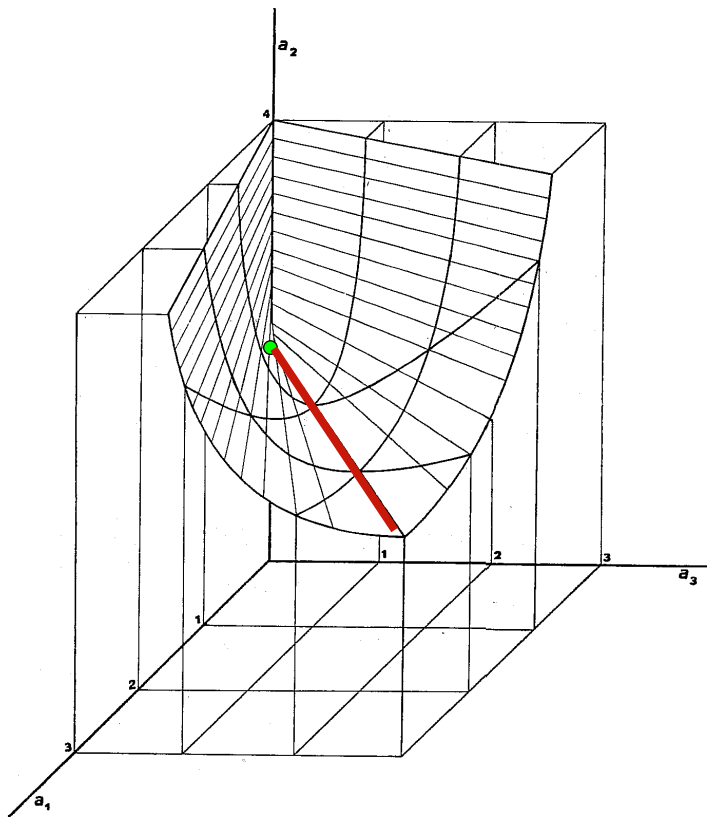
$$r = 1, \quad i.e. \quad a_3 = a_1$$

Oene BOTTEMA The Routh-Hurwitz condition for the biquadratic equation *Indagationes Mathematicae*, 18 (1956) 403-406

Pioneering works by Bottema (1956)

$$a_2 = 2 + \frac{(a_1 - a_3)^2}{a_1 a_3} > 2, \quad a_3 \neq a_1$$

$$a_2 = 2, \quad a_3 = a_1$$



„Here is the discontinuity we mentioned above. It plays a part in questions regarding the stability of equilibrium.

The coefficients a_1 and a_3 depend on the linear damping forces and it is well known that the stability condition may change in a discontinuous way if a very small damping vanishes at all.

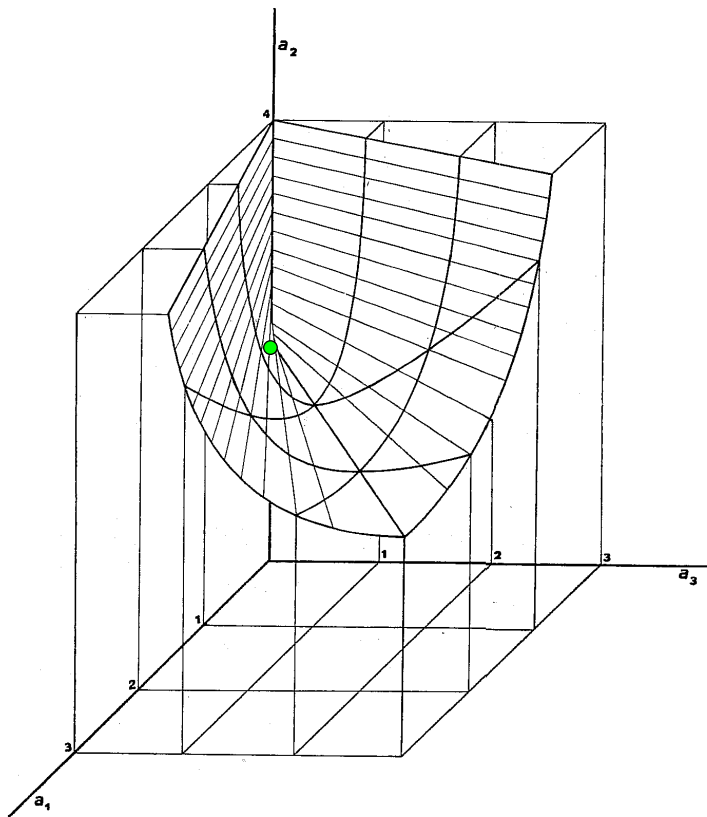
The phenomenon may be illustrated by a geometrical diagram.” Bottema, 1956

Oene BOTTEMA The Routh-Hurwitz condition for the biquadratic equation *Indagationes Mathematicae*, 18 (1956) 403-406

Generic singularities by Arnold (1971)

Double imaginary
root at the singular point:

$$\lambda = i$$



V. I. Arnol'd

85

two-parameter families.¹ These singularities can be listed to within diffeomorphism as follows:

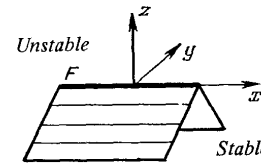


Fig. 4.14.

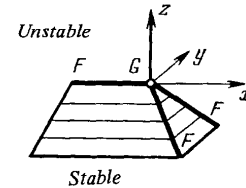


Fig. 4.15.

Two faces meeting along a ridge (F_1 ; Fig. 4.14): $z + |y| = 0$.

Three faces meeting at a corner ($G_{3,4,5}$; Fig. 4.15): $z + \max(x, |y|) = 0$.

Cuspidal point on a ridge (G_2 ; Fig. 4.17): $z + |\operatorname{Re}\sqrt{(x + iy)}| = 0$.

(This surface in $\mathbb{R}^3$ is diffeomorphic to that given by the equations $XY^2 = Z^2$, where $Y \geq 0$.)

Node on a ridge (G_1 ; Fig. 4.16): $z + \lambda(x, y) = 0$, where λ is the greatest real part of the roots of the equation $\lambda^3 = x\lambda + y$. (This surface in $\mathbb{R}^3$ is diffeomorphic to that given by $X^2Y^2 = Z^2$, $X \geq 0$, $Y \geq 0$.)

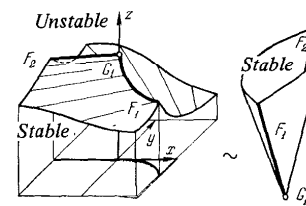


Fig. 4.16.

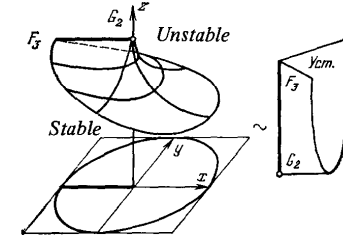
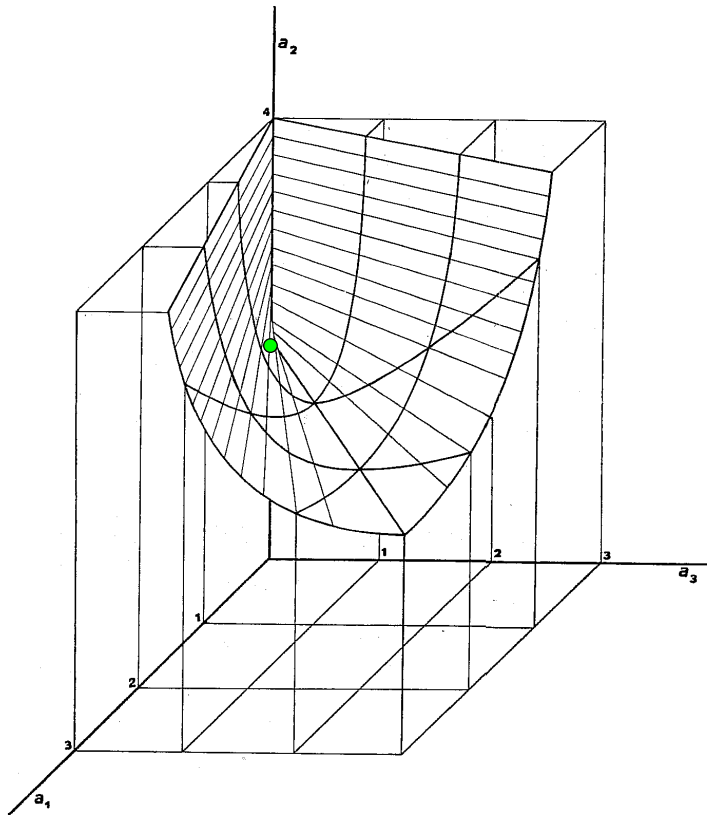


Fig. 4.17.

The acute angles of the stability boundary always point into the domain of instability.

Whitney umbrella singularity at $a_2=2$, $a_1=a_3=0$

Double imaginary
root at the singular point:
 $\lambda=i$



$$\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + 1$$

Companion matrix:

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -a_3 & -a_2 & -a_1 \end{pmatrix}$$

Jordan form:

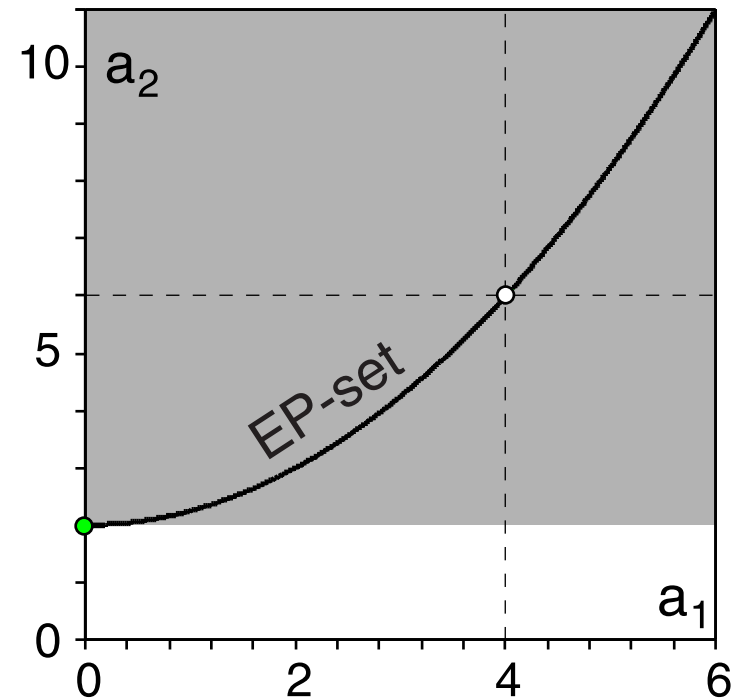
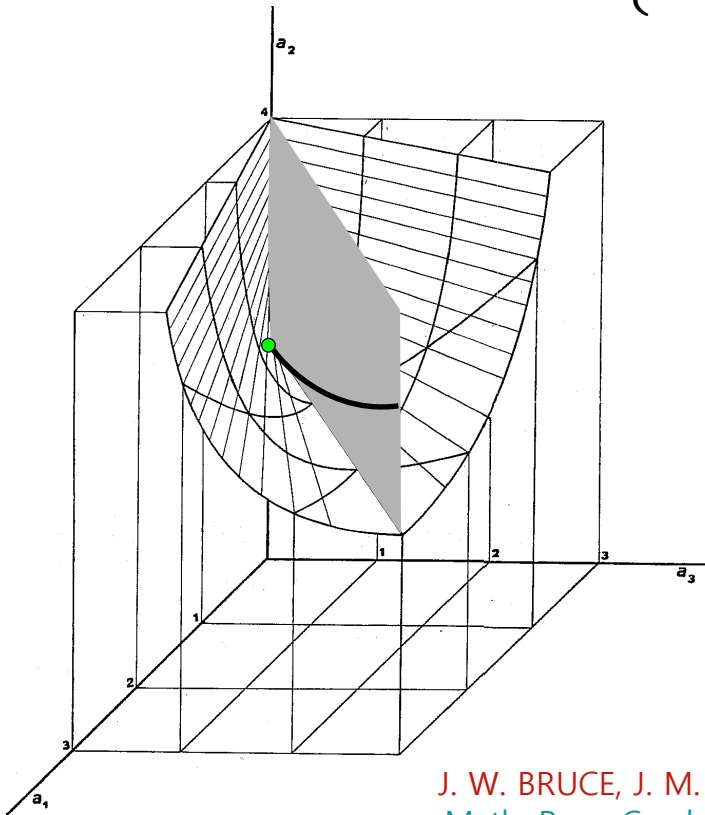
$$\mathbf{J}_2 = \begin{pmatrix} i & 1 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 1 \\ 0 & 0 & 0 & -i \end{pmatrix}$$

V. I. ARNOLD On Matrices Depending on Parameters [Russ. Math. Surv. 26 \(1971\) 29-43](#)

Tangent cone to the stability domain

Tangent cone: $\{(a_1, a_3, a_2) : a_1 = a_3, a_1 > 0, a_2 > 2\}$

EP-set: $\left\{ (a_1, a_3, a_2) : a_1 = a_3, a_2 = 2 + \frac{a_1^2}{4} \right\}$



J. W. BRUCE, J. M. WEST Functions on a crosscap.
Math. Proc. Camb. Philos. Soc. 123 (1998) 19–39.

Tangent cone to the stability domain

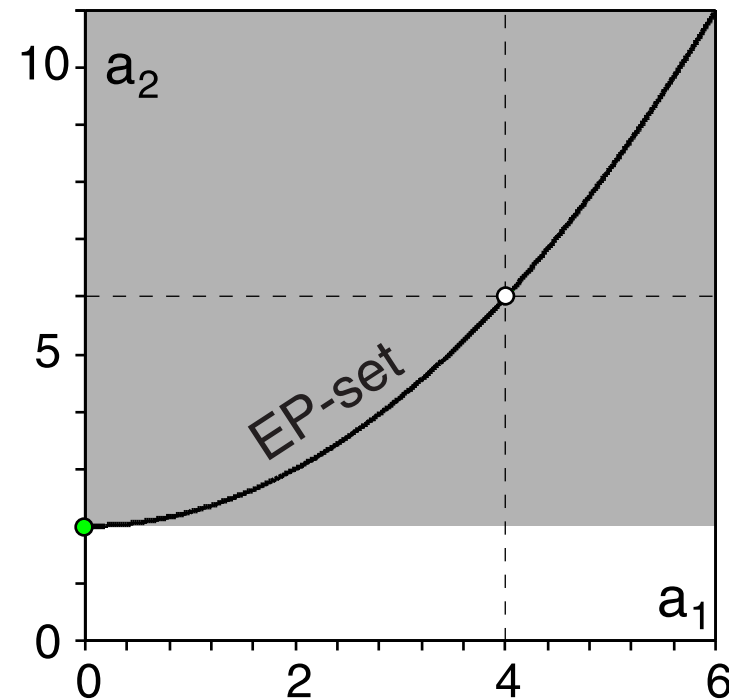
Tangent cone: $\{(a_1, a_3, a_2) : a_1 = a_3, a_1 > 0, a_2 > 2\}$

EP-set: $\left\{ (a_1, a_3, a_2) : a_1 = a_3, a_2 = 2 + \frac{a_1^2}{4} \right\}$

Multiple eigenvalues
at the EP-set:

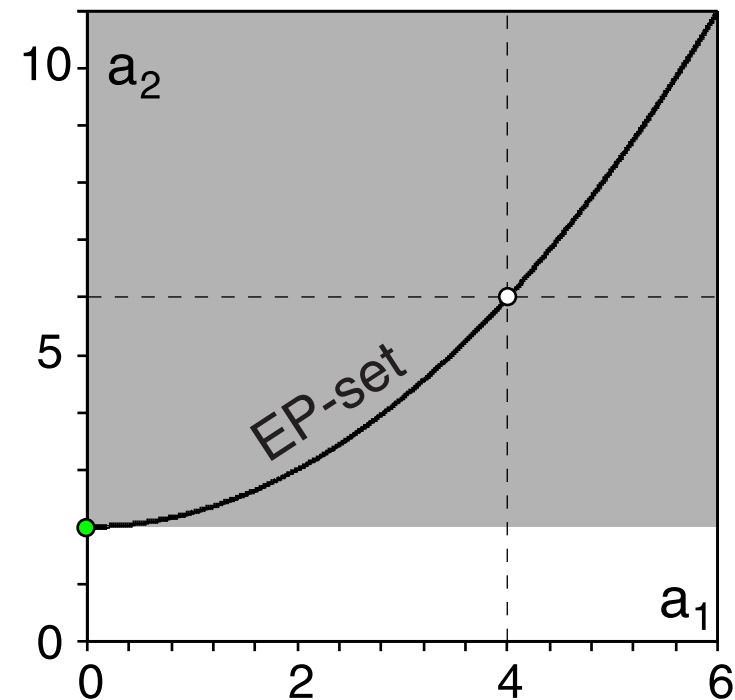
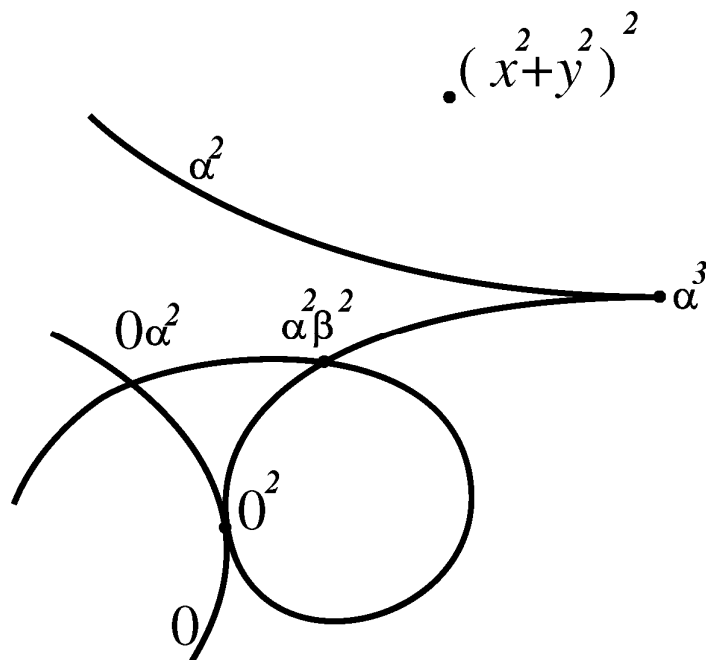
$$\lambda_1 = \lambda_2 = -\frac{a_1}{4} - \frac{1}{4}\sqrt{a_1^2 - 16}$$

$$\lambda_3 = \lambda_4 = -\frac{a_1}{4} + \frac{1}{4}\sqrt{a_1^2 - 16}$$



Bifurcation diagrams of families of real matrices

Double eigenvalues $x+iy$ with the Jordan block:
codim=2



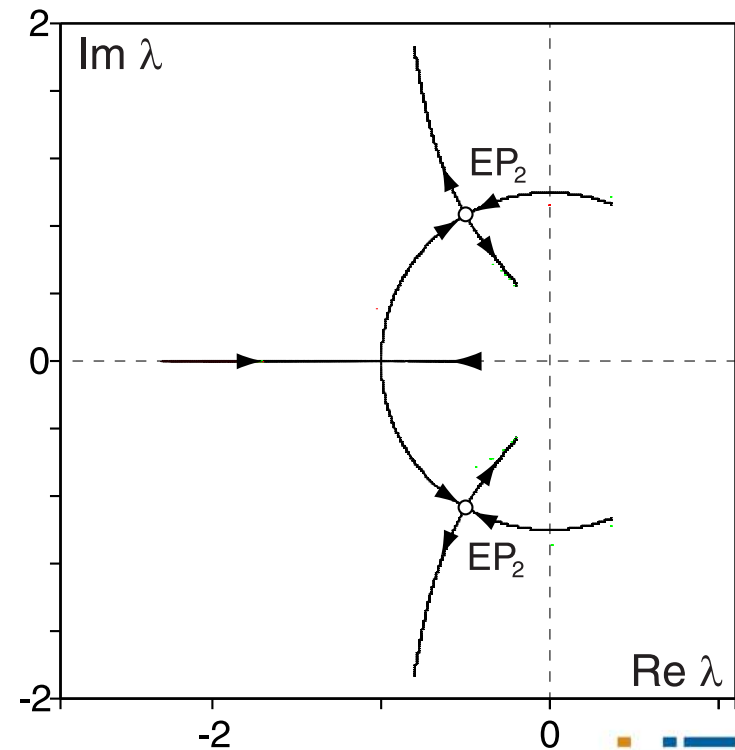
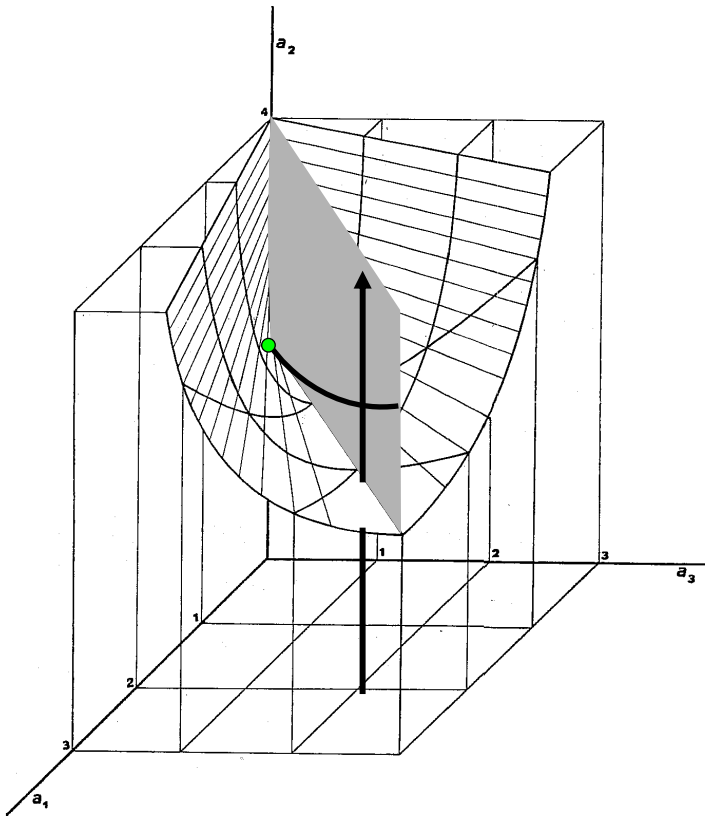
D.M. GALIN Real matrices depending on parameters. *Uspekhi Mat. Nauk* 27 (1972) 241–241.

Movement of eigenvalues

Parameters change within the tangent cone

$$a_1 = a_3 = 2, \quad 0 \leq a_2 \leq 6$$

Collisions on the unit circle at exceptional points (EP_2), $\text{Re}\lambda < 0$

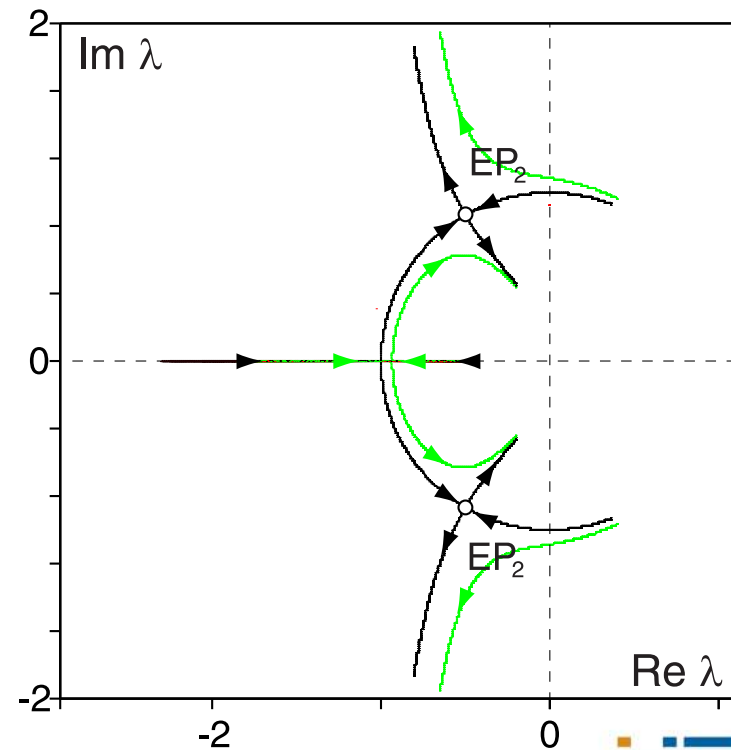
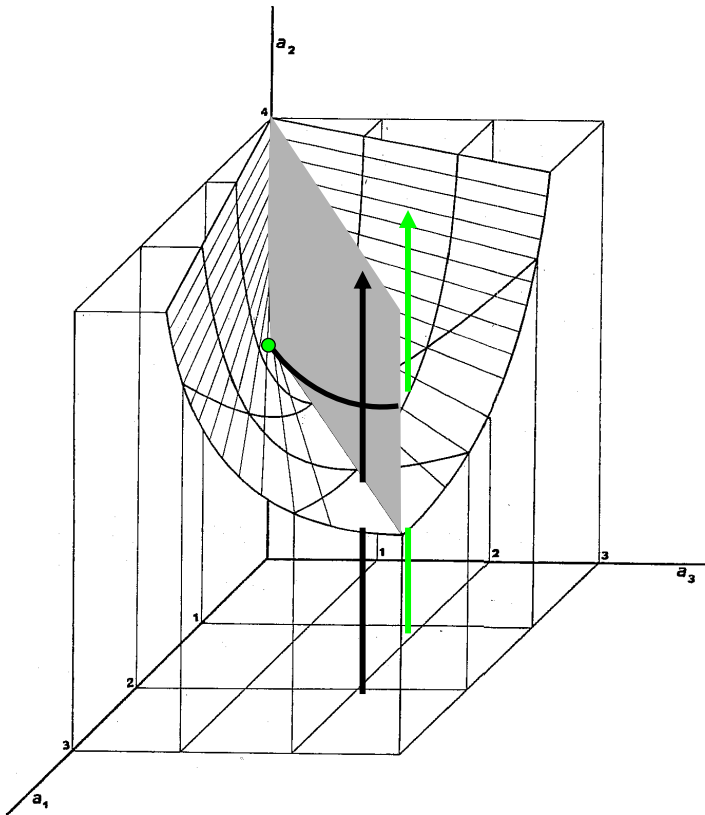


Imperfect merging of modes

Parameters change right to the tangent cone

$$a_1 = 1.7, \quad a_3 = 2, \quad 0 \leq a_2 \leq 6$$

Avoided crossings near EP_2 , $\text{Re}\lambda < 0$

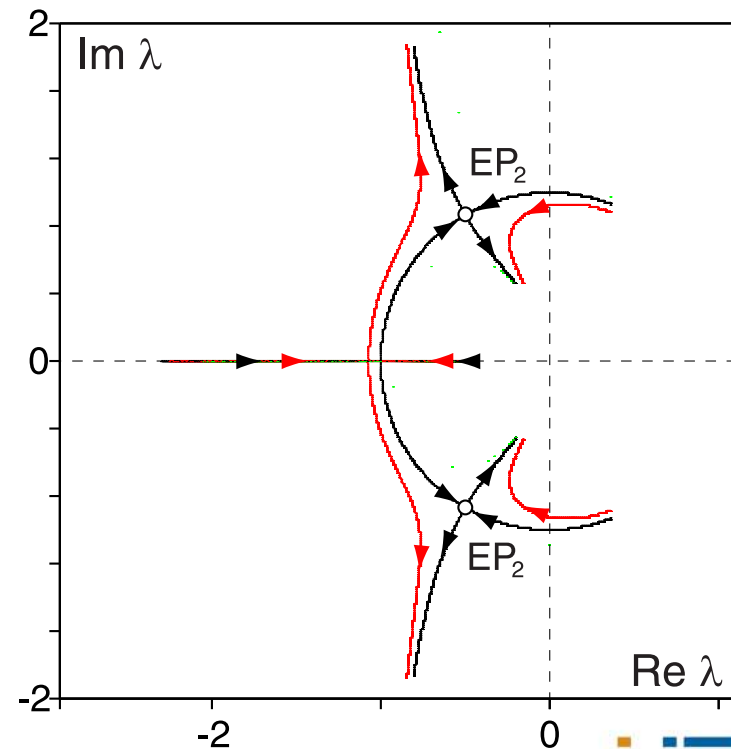
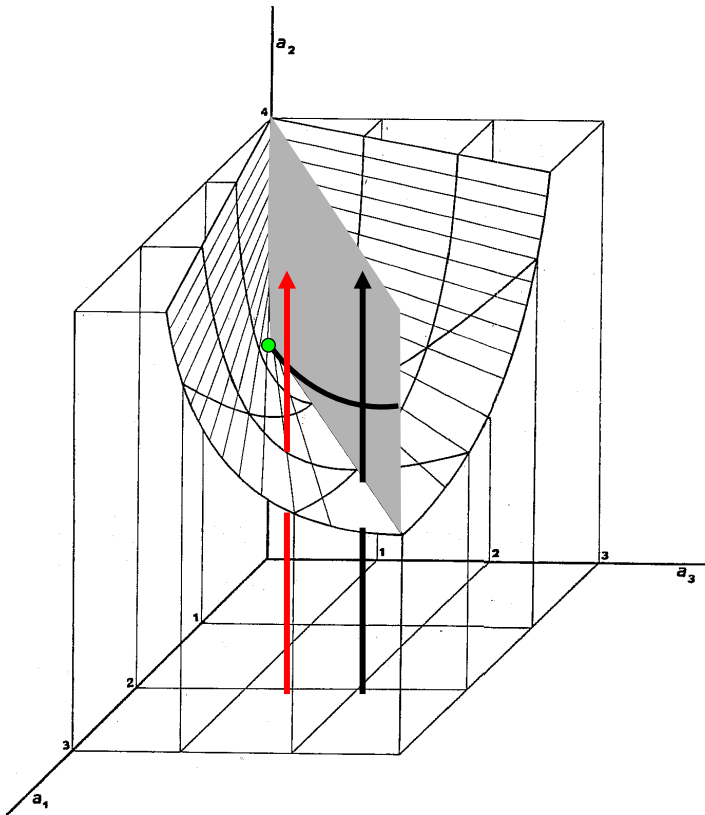


Exchange of instability between branches

Parameters change left to the tangent cone

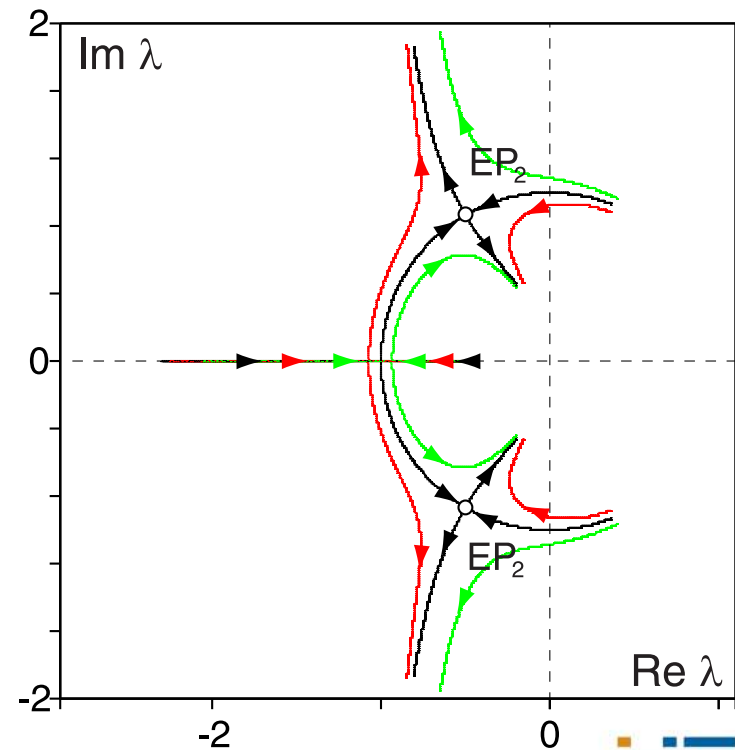
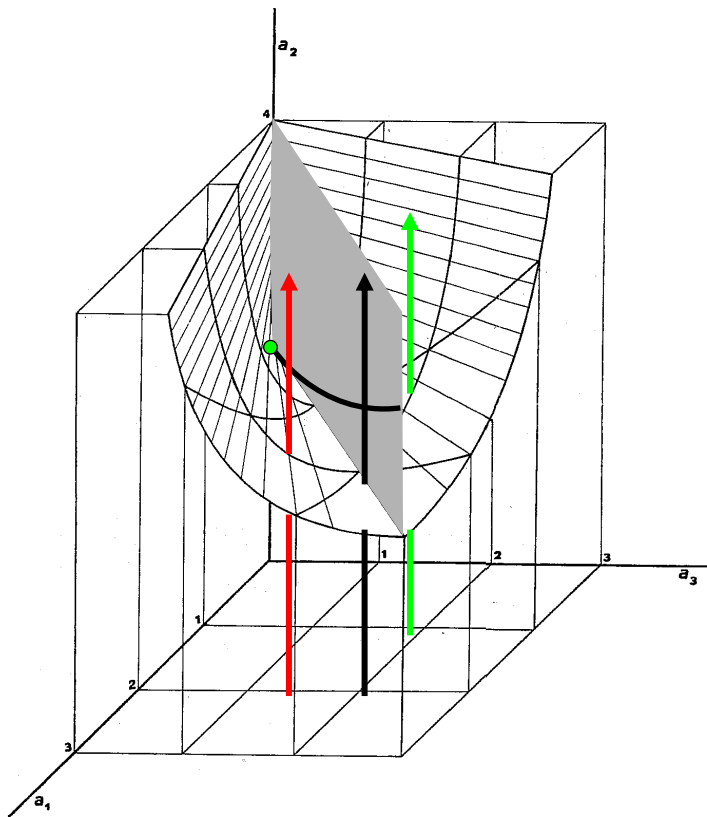
$$a_1 = 2, \quad a_3 = 1.7, \quad 0 \leq a_2 \leq 6$$

Avoided crossings near EP_2 , $\text{Re} \lambda < 0$



Selective role of the tangent cone

It determines which mode is destabilized by dissipation
because
the set of multiple complex eigenvalues (EP-set) is within it



Approximation of the eigenvalue trajectories

$$p = \lambda^4 + a_1 \lambda^3 + a_2 \lambda^2 + a_3 \lambda + 1$$

A double eigenvalue at the EP-set: $a_{1,0} = a_{3,0}$, $a_{2,0} = 2 + \frac{a_{1,0}^2}{4}$

$$\lambda_0 = -\frac{a_{1,0}}{4} + \frac{i}{4} \sqrt{16 - a_{1,0}^2}$$

Perturbation of parameters:

$$a_1 = a_{1,0} + \varepsilon \hat{a}_1, \quad a_2 = a_{2,0} + \varepsilon \hat{a}_2, \quad a_3 = a_{3,0} + \varepsilon \hat{a}_3$$

Newton-Puiseux series: $\lambda = \lambda_0 + \varepsilon^{1/2} \lambda_1 + \varepsilon \lambda_2 + \dots$

$$\lambda_1^2 = -2(\hat{a}_1 \partial_{a_1} p + \hat{a}_2 \partial_{a_2} p + \hat{a}_3 \partial_{a_3} p)(\partial_{\lambda}^2 p)^{-1}$$

J. MORO et al. On the Lidskii-Vishik-Lyusternik perturbation theory for eigenvalues of matrices with arbitrary Jordan structure [SIAM J. Matrix Anal. Appl. 18 \(1997\) 793-817](#)

Approximation of the eigenvalue trajectories

Rewrite the perturbation formula to emphasize multiple parameters:

$$\operatorname{Re}\Delta\lambda + i\operatorname{Im}\Delta\lambda = \pm\sqrt{\langle\mathbf{f}, \Delta\mathbf{a}\rangle + i\langle\mathbf{g}, \Delta\mathbf{a}\rangle} + O(\|\Delta\mathbf{a}\|)$$

$$\Delta\lambda = \lambda - \lambda_0 \quad \Delta\mathbf{a} = \varepsilon(\hat{a}_1, \hat{a}_2, \hat{a}_3)$$

$$\mathbf{f} = (f_1, f_2, f_3) \quad \mathbf{g} = (g_1, g_2, g_3)$$

$$\gamma := (f_1g_2 - f_2g_1)\Delta a_1 + (f_3g_2 - f_2g_3)\Delta a_3$$

Compute the coefficients at the EP-set:

$$f_2 = \frac{8 - a_{1,0}^2}{2(a_{1,0}^2 - 16)}, \quad g_2 = \frac{a_{1,0}\sqrt{16 - a_{1,0}^2}}{2(a_{1,0}^2 - 16)}, \quad \gamma = \frac{4\sqrt{16 - a_{1,0}^2}}{(16 - a_{1,0}^2)^2}(\Delta a_3 - \Delta a_1)$$

Approximation of the eigenvalue trajectories

Trajectories of eigenvalues in the complex plane
near an EP_2 point as a_2 varies:

$$g_2(\operatorname{Re}\Delta\lambda)^2 - 2f_2\operatorname{Re}\Delta\lambda\operatorname{Im}\Delta\lambda - g_2(\operatorname{Im}\Delta\lambda)^2 = \frac{4\sqrt{16 - a_{1,0}^2}}{(16 - a_{1,0}^2)^2}(\Delta a_3 - \Delta a_1)$$

If $\Delta a_3 = \Delta a_1$, i.e. perturbation is within the tangent cone,
then the eigenvalues merge at the 1:1 resonance

If $\Delta a_3 \neq \Delta a_1$, i.e. perturbation is outside the tangent cone,
then eigenvalues move along the branches of a hyperbola
and demonstrate an imperfect merging

Stability boundary via perturbation of eigenvalues

$$p = \lambda^4 + a_1 \lambda^3 + a_2 \lambda^2 + a_3 \lambda + 1$$

In the 'ideal' case ($a_1 = a_3 = 0$) : $\lambda_{id}^2 = -\frac{a_2}{2} \pm \frac{1}{2} \sqrt{a_2^2 - 4}$

Perturbation of a simple imaginary eigenvalue at $a_2 > 0$

$$\lambda = \lambda_{id} - a_1 \frac{\partial_{a_1} p}{\partial_{\lambda} p} - a_3 \frac{\partial_{a_3} p}{\partial_{\lambda} p} + o(a_1, a_3)$$

Approximating stability boundary by vanishing the linear increment:

$$a_1 \lambda_{id}^2(a_2) + a_3 = 0$$

The approximate boundary yields exactly the Hurwitz threshold:

$$a_2 = 2 + \frac{(a_1 - a_3)^2}{a_1 a_3}, \quad \lim_{\varepsilon \rightarrow 0} \frac{(\varepsilon \hat{a}_1 - \varepsilon \hat{a}_3)^2}{\varepsilon \hat{a}_1 \varepsilon \hat{a}_3} - 2 = O(1)$$

Same ideas work in a more complicated case

Gyroscopic stabilization of non-conservative systems

$$\ddot{x} + (\gamma G + \delta D)\dot{x} + (\nu N + P)x = 0, \quad x \in R^m, \quad m \text{ is even}$$

$$G = -G^T, \quad D = D^T, \quad N = -N^T, \quad P = P^T = -K < 0$$

Statically unstable potential system $\ddot{x} + Px = 0$

can be gyroscopically stabilized, i.e. $\ddot{x} + \gamma G\dot{x} + Px = 0$

is stable if $\det G \neq 0$ and $|\gamma| > \gamma_0 > 0$

Do the non-conservative perturbations δD and νN preserve the gyroscopic stabilization?

Same ideas work in a more complicated case

A double eigenvalue with the Jordan chain at the threshold of gyroscopic stabilization

$$(-I\omega_0^2 + i\omega_0\gamma_0 G - K)u_0 = 0$$

$$(-I\omega_0^2 + i\omega_0\gamma_0 G - K)u_1 = -(2i\omega_0 I + \gamma_0 G)u_0$$

Orthogonality: $u_0^*(2i\omega_0 I + \gamma_0 G)u_0 = 0$

Rayleigh quotient: $\omega_0^2 = \frac{u_0^* K u_0}{u_0^* u_0}$

Hamiltonian-Hopf bifurcation

$$i\omega(\gamma) = i\omega_0 \pm i\mu\sqrt{\gamma - \gamma_0} + O(\gamma - \gamma_0)$$

$$u(\gamma) = u_0 \pm i\mu u_1 \sqrt{\gamma - \gamma_0} + O(\gamma - \gamma_0)$$

$$\mu^2 = -\frac{2\omega_0^2 u_0^* u_0}{\gamma_0(\omega_0^2 u_1^* u_1 + u_1^* K u_1 - i\omega_0 \gamma_0 u_1^* G u_1 - u_0^* u_0)}$$

Same ideas work in a more complicated case

Non-conservative perturbation of simple eigenvalues at $\gamma > \gamma_0$

$$\lambda = i\omega + \frac{\omega^2 u^* Du \delta - i\omega u^* Nu \nu}{u^* Ku - \omega^2 u^* u} + o(\delta, \nu)$$

Stability boundary: $\nu = \beta(\gamma)\delta, \quad \beta(\gamma) = -i\omega(\gamma) \frac{u^*(\gamma) Du(\gamma)}{u^*(\gamma) Nu(\gamma)}$

New threshold of gyroscopic stability:

$$\gamma_{cr}(\beta) = \gamma_0 + \frac{n_1^2(\beta - \beta_0)^2}{\mu^2(\omega_0 d_2 - \beta_0 n_2 - d_1)^2}$$

Valid at: $|\beta - \beta_0| \ll 1, \quad \beta_0 = \beta(\gamma_0) = -i\omega_0 \frac{u_0^* Du_0}{u_0^* Nu_0}$

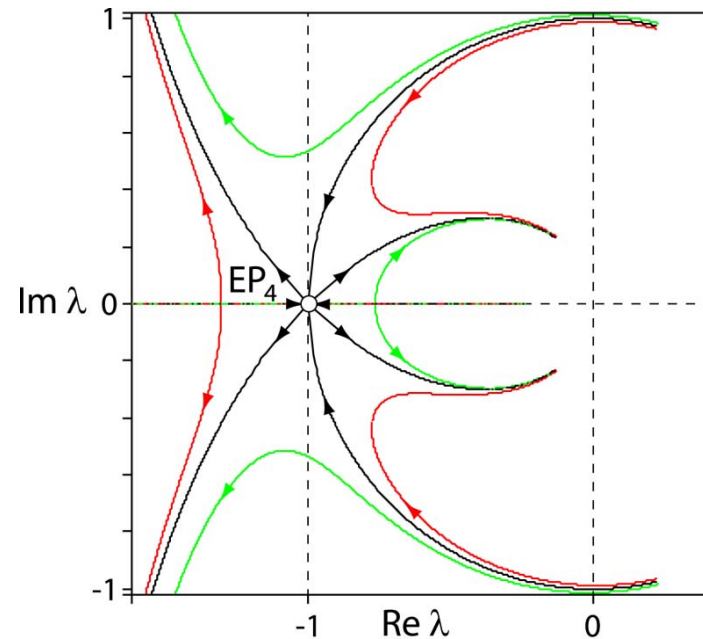
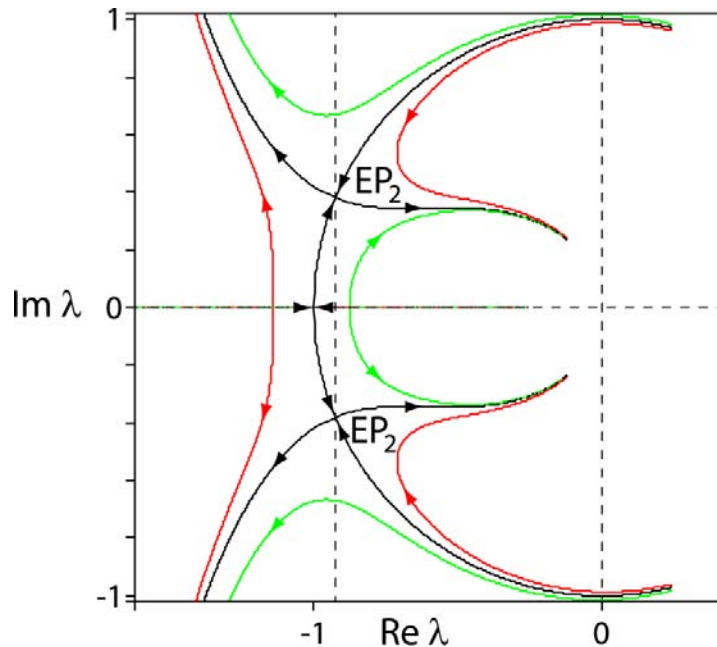
$$d_1 = \operatorname{Re}(u_0^* Du_0), \quad d_2 = \operatorname{Im}(u_0^* Du_1 - u_1^* Du_0)$$

$$n_1 = \operatorname{Im}(u_0^* Nu_0), \quad n_2 = \operatorname{Re}(u_0^* Nu_1 - u_1^* Nu_0)$$

Robust stabilizatton by minimizing the root abscissa

$$p = \lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + 1 \quad \mathbf{a}(p) = \max\{\operatorname{Re}\lambda : p(\lambda) = 0\}$$

$$\mathbf{a}^* = \inf_{a_1, a_2, a_3} \mathbf{a}(p)$$



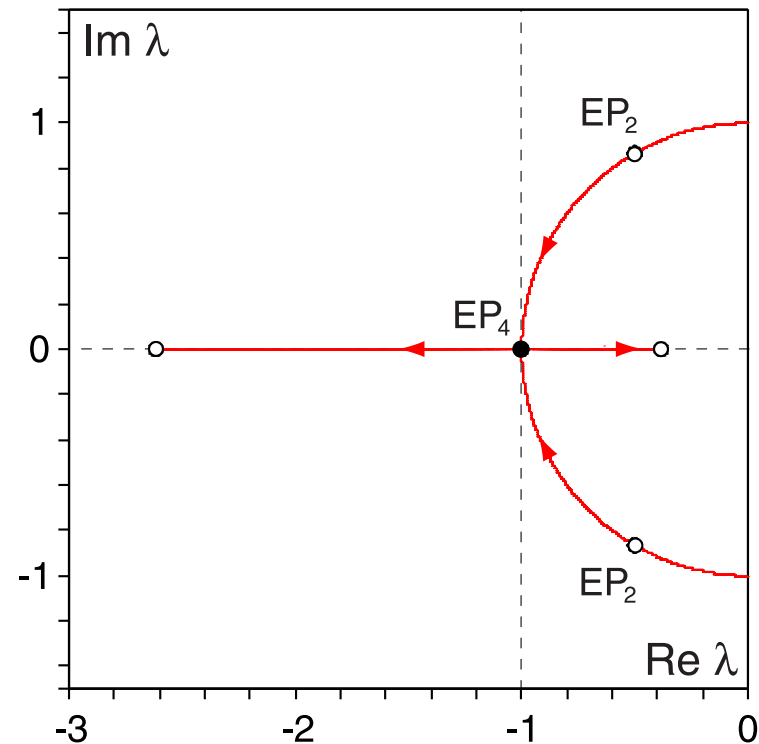
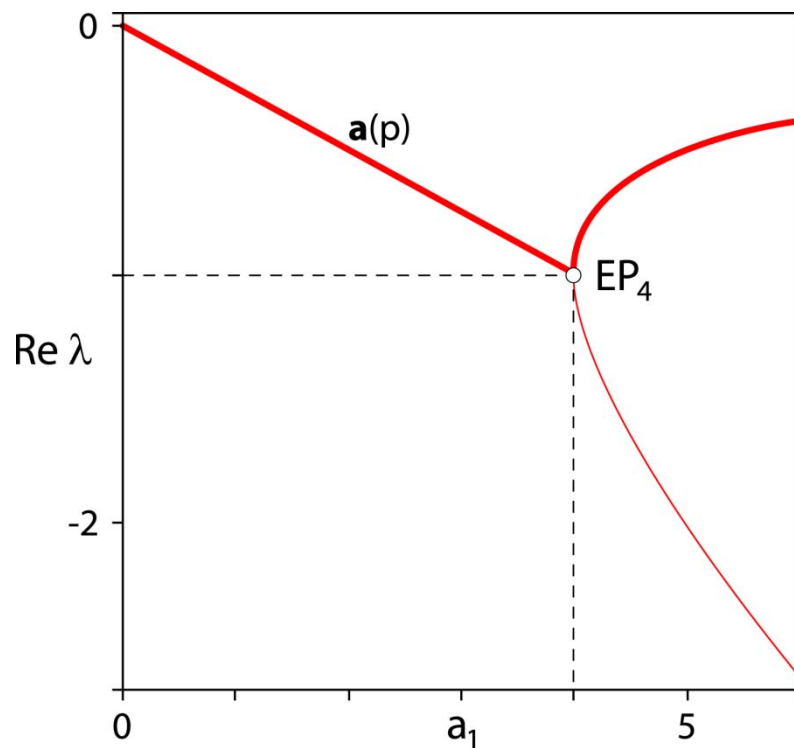
The minimizer seems to be at the EP-set

Robust stabilization by minimizing the root abscissa

Multiple eigenvalues at the EP-set

$$\lambda_1 = \lambda_2 = -\frac{a_1}{4} - \frac{1}{4}\sqrt{a_1^2 - 16}$$

$$\lambda_3 = \lambda_4 = -\frac{a_1}{4} + \frac{1}{4}\sqrt{a_1^2 - 16}$$

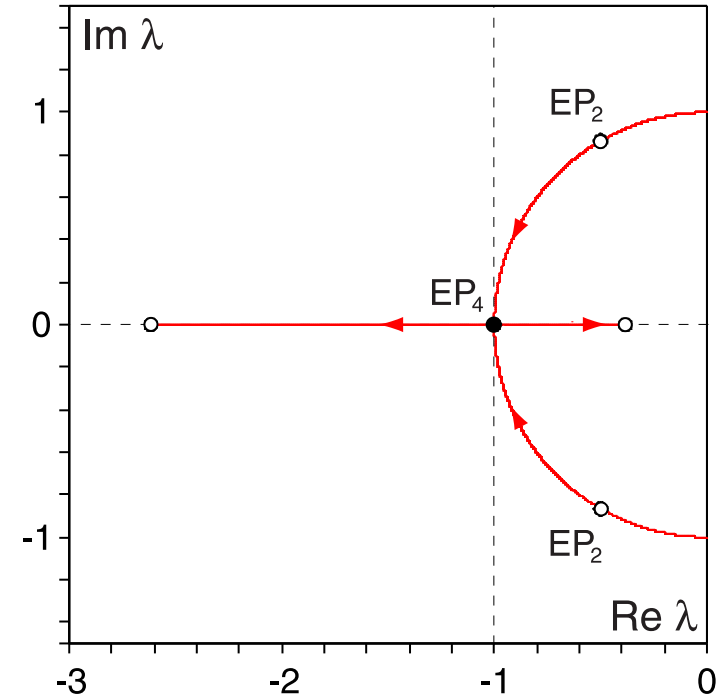
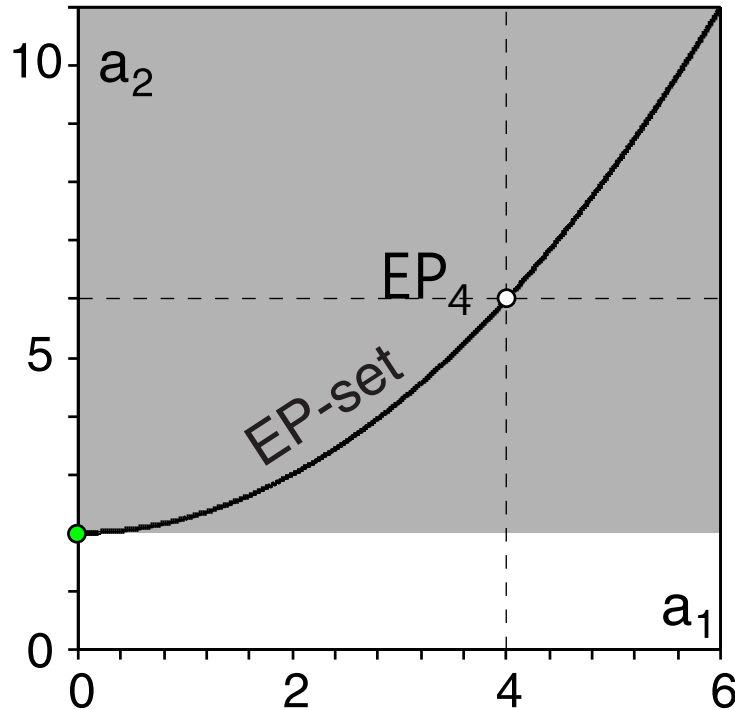


O. N. KIRILLOV, M. L. OVERTON Robust stability at the swallowtail singularity
Frontiers in Physics, 1 (2013) 24

Robust stabilizaton by minimizing the root abscissa

$$\text{EP}_4 : \quad \lambda = -1, \quad a_1 = 4, \quad a_2 = 6, \quad a_3 = 4$$

The minimizer $p^*(\lambda) = \lambda^4 + 4\lambda^3 + 6\lambda^2 + 4\lambda + 1 = (\lambda + 1)^4$



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Robust stabilizaton by minimizing the root abscissa

Th. (Blondel et al. 2012): Given a family of real polymomials and a single affine constraint

$$P = \{\lambda^n + a_1\lambda^{n-1} + \dots + a_{n-1}\lambda + a_n : b_0 + \sum_{j=1}^n b_j a_j = 0\}$$

Let $h(\lambda) = b_n\lambda^n + b_{n-1}C_n^{n-1}\lambda^{n-1} + \dots + b_1C_n^1\lambda + b_0$

$$k = \max\{j : b_j \neq 0\}$$

Then

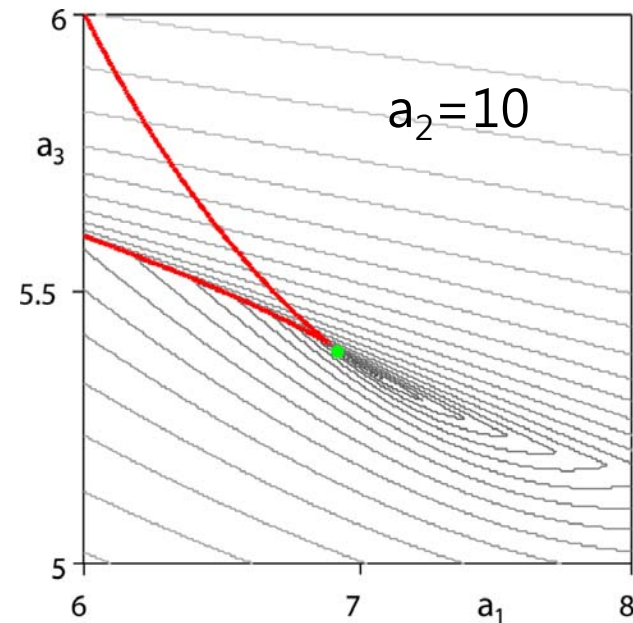
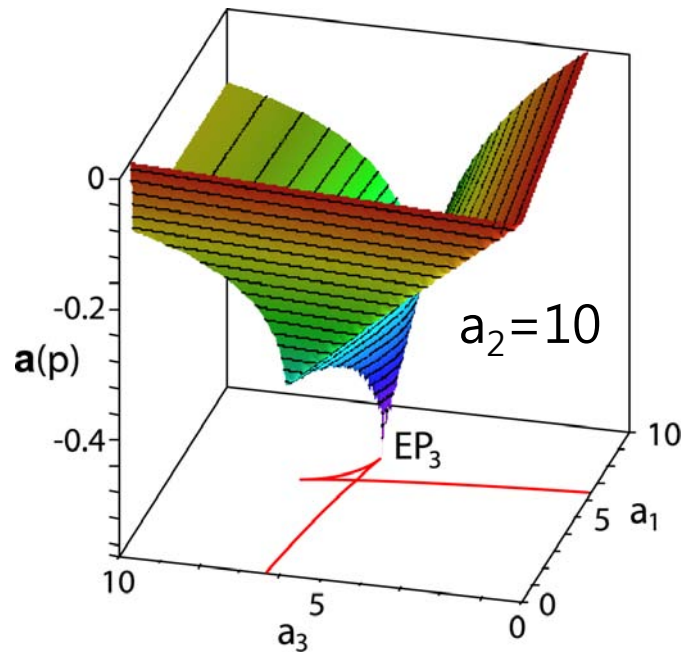
$$\mathbf{a}^* := \inf_{p \in P} \mathbf{a}(p) = -\max\{\zeta \in R : h^{(i)}(\zeta) = 0 \text{ for some } i \in \{0, \dots, k-1\}\}$$

The minimizing polynomial $p^*(\lambda) = (\lambda - \gamma)^n \in P$, $\gamma = \mathbf{a}^*$
if and only if $-\mathbf{a}^*$ is a root of h .

Robust stabilizaton by minimizing the root abscissa

$$p = \lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + 1$$

Additional constraint ($a_2=\text{const}$) reduces the root multiplicity from 4 to 3 at the minimum

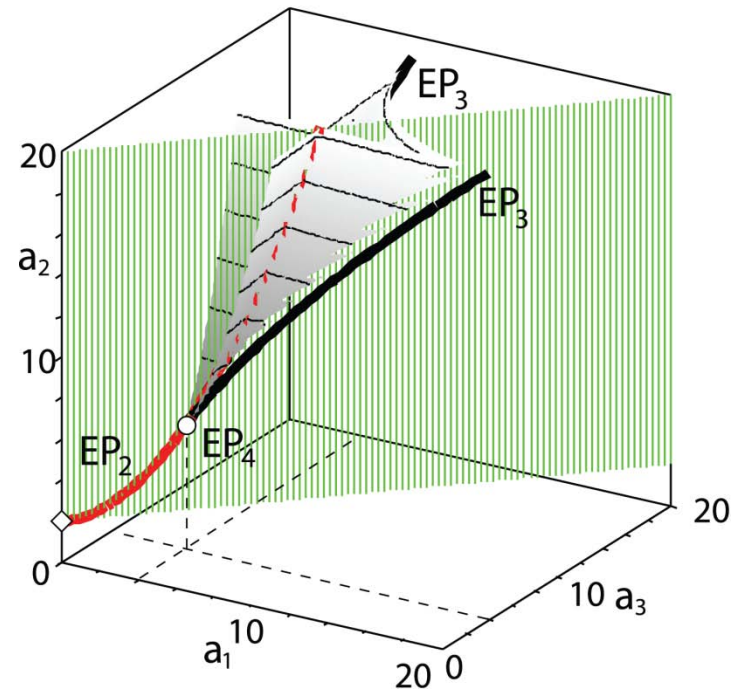
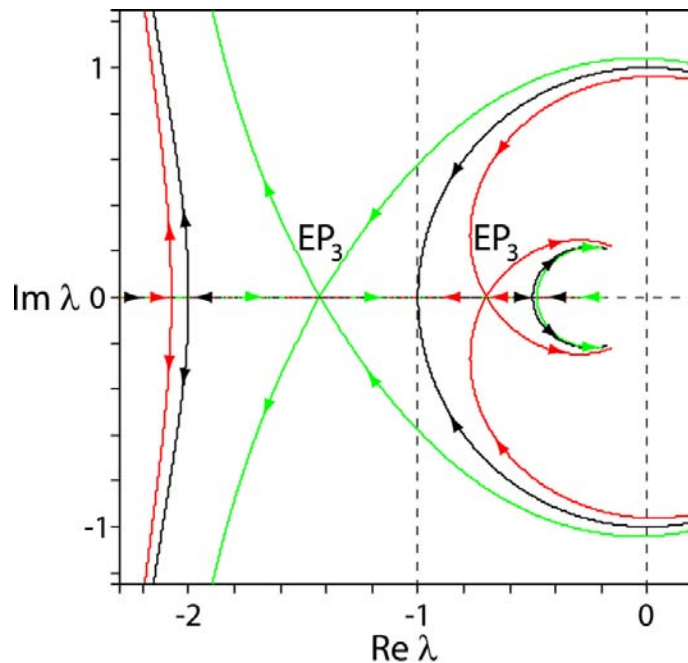


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Minima of the root abscissa and discriminant

$$p = \lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + 1$$

Discriminant of the polynomial has a Swallowtail singularity at EP_4 and cuspidal edges at EP_3

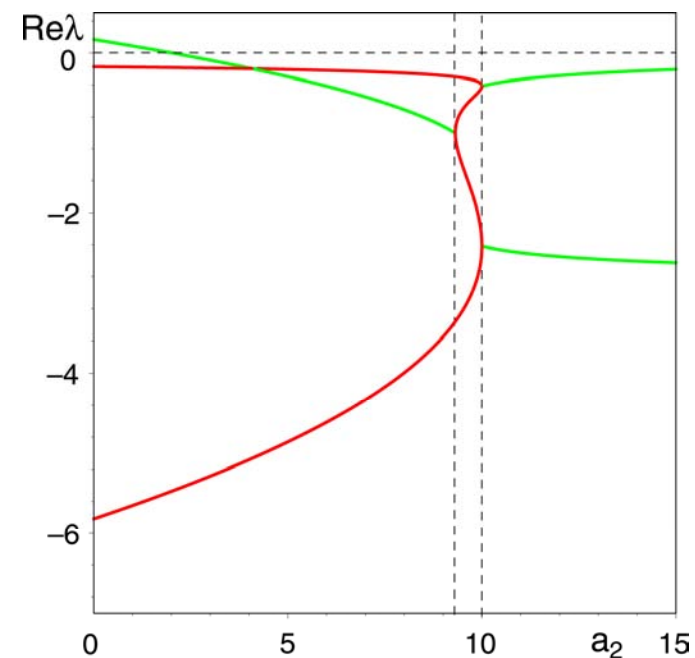
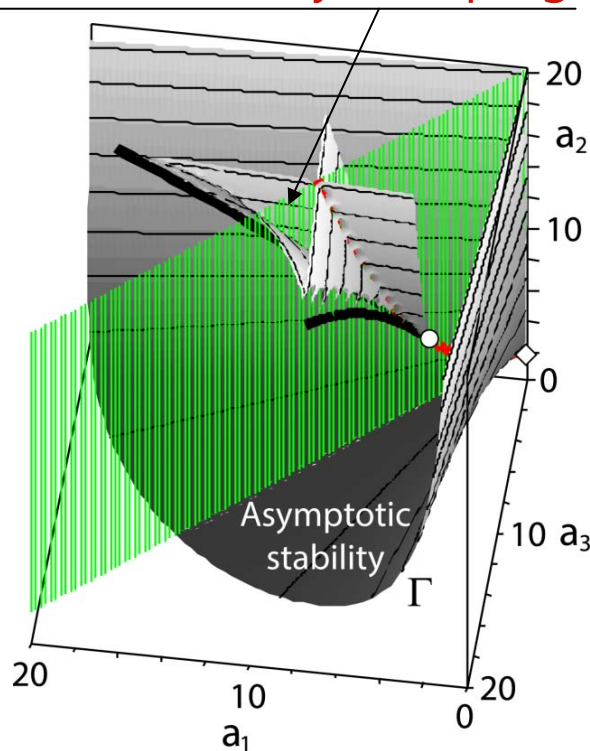


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Asymptotic stability and heavy damping

All the roots are real and simple inside the „spike“ (heavy damping).

Root abscissa minimization yields the points at the boundary of the domain of heavy damping.



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Asymptotic stability and heavy damping

$$\ddot{x} + D\dot{x} + Kx = 0, \quad x \in R^m$$

In a heavily damped system all the eigenvalue of $L(\lambda) = \lambda^2 + D\lambda + K$ are semi-simple real and negative

Spectral abscissa of the matrix polynomial $L(\lambda)$

$$\alpha(D, K) = \mathbf{a}(\det(L(\lambda))) = \max\{\operatorname{Re}\lambda : \det(L(\lambda)) = 0\}$$

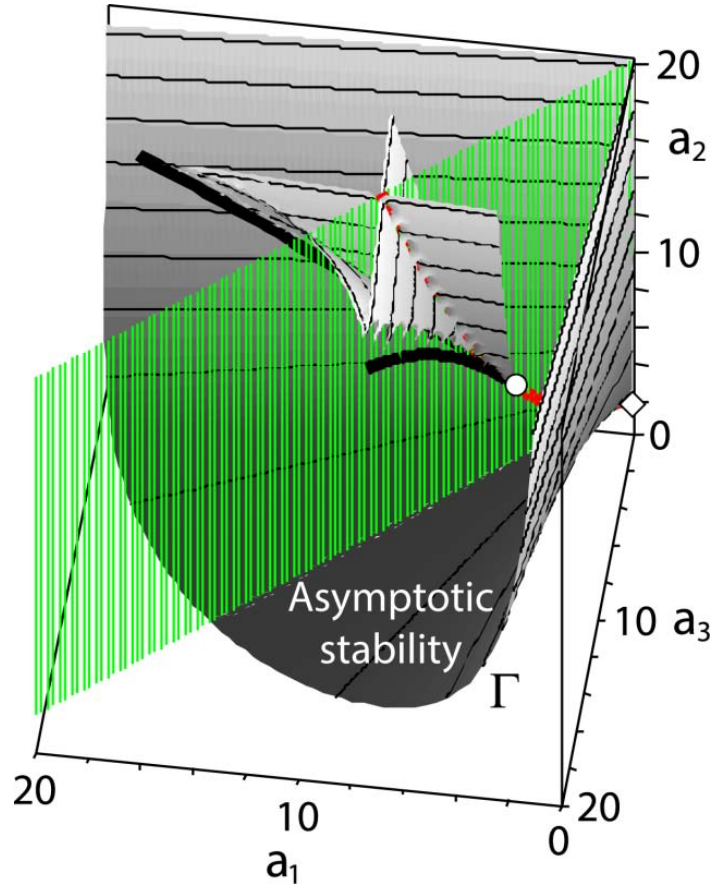
$$\alpha^* = \inf_D \alpha(D, K)$$

Freitas and Lancaster (1999) $\forall K = K^T > 0 : \quad \alpha^* \geq -\sqrt[2m]{\det K};$

$\alpha^* = -\sqrt[2m]{\det K}$ iff $L(\lambda)$ has an eigenvalue of multiplicity $2m$

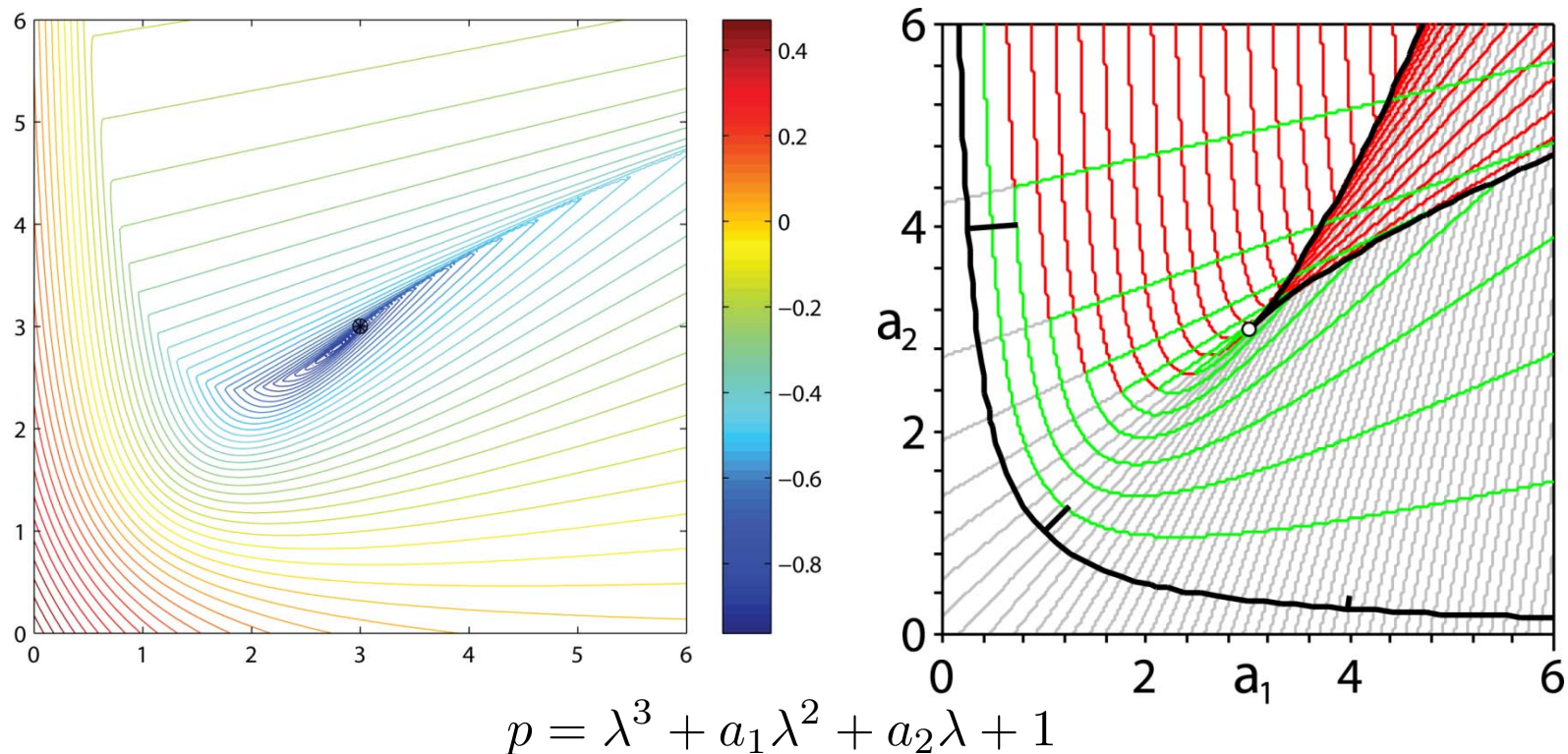
P. FREITAS, P. LANCASTER On the optimal value of the spectral abscissa for a system of linear oscillators *SIAM J Matrix Anal. Appl.* 21 (1999) 195–208

Robust stability at the Swallowtail



1. Multiple eigenvalues mysteriously originate in various contexts of optimizing stability
2. They are related to singularities on the stability boundary and the discriminant set
3. Why optimization of abscissa so naturally tends to the most degenerate eigenvalue?

Does optimizing the abscissa possesses a physical interpretation in terms of wave front propagation with multiple roots corresponding to caustics?



V. I. ARNOLD Lectures on bifurcations in versal families [Russ. Math. Surv. 27 \(1972\) 54–123](#)

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14

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Oleg N. Kirillov

NONCONSERVATIVE STABILITY PROBLEMS OF MODERN PHYSICS

Oleg N. Kirillov
NONCONSERVATIVE STABILITY PROBLEMS OF MODERN PHYSICS

This work gives a complete overview on the subject of nonconservative stability from the modern point of view. Relevant mathematical concepts are presented, as well as rigorous stability results and numerous classical and contemporary examples from mechanics and physics.

It deals with both finite- and infinite-dimensional nonconservative systems and covers the fundamentals of the theory, including such topics as Lyapunov stability and linear stability analysis, Hamiltonian and gyroscopic systems, reversible and circulatory systems, influence of structure of forces on stability, and dissipation-induced instabilities, as well as concrete physical problems, including perturbative techniques for nonself-adjoint boundary eigenvalue problems, theory of the destabilization paradox due to small damping in continuous circulatory systems, Krein-space related perturbation theory for the MHD kinematic mean field α -dynamo, analysis of Campbell diagrams and friction-induced flutter in gyroscopic continua, non-Hermitian perturbation of Hermitian matrices with applications to optics, and magnetorotational instability and the Velikhov-Chandrasekhar paradox.

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