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Determining role of Krein signature for three-dimensional Arnold tongues of oscillatory dynamos

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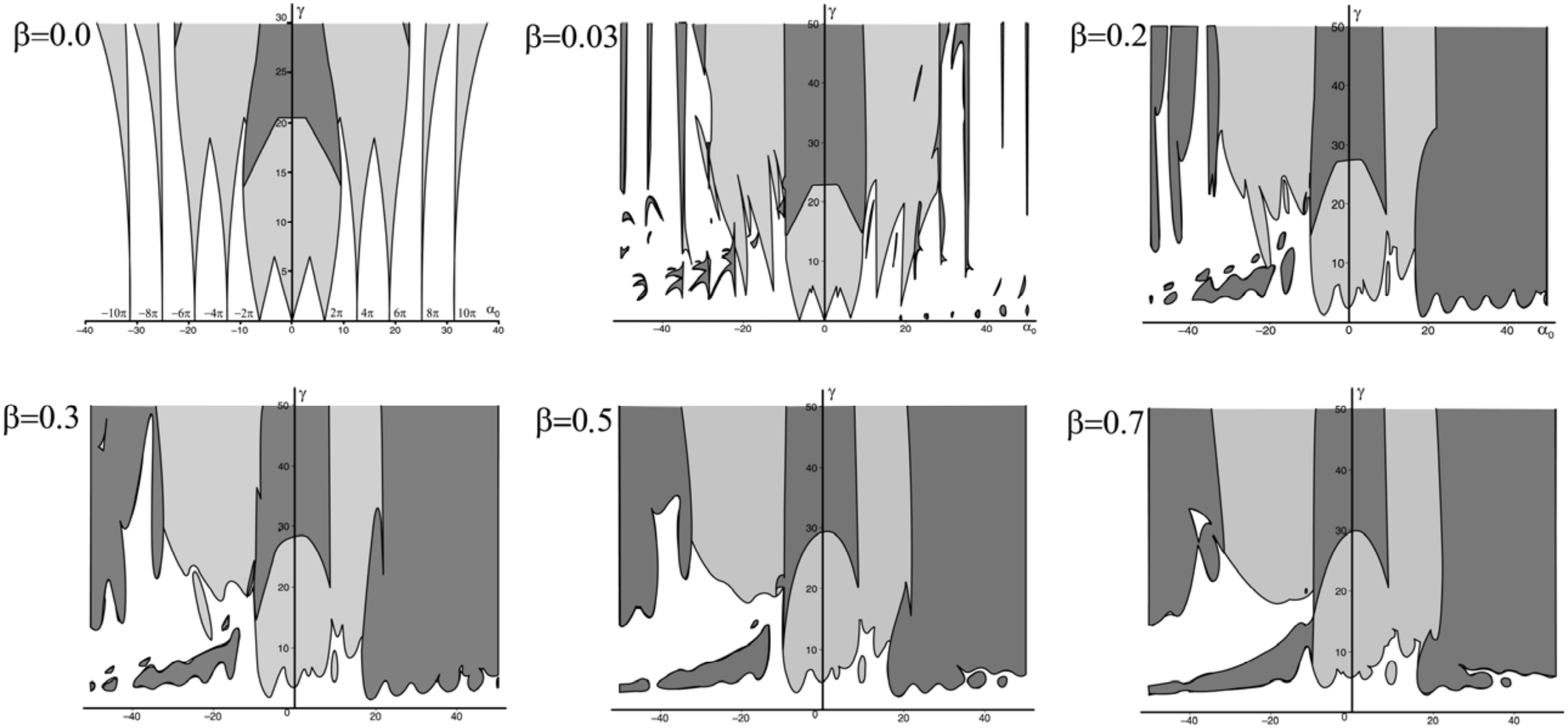
MHD α^2 – dynamo with the variation of the boundary conditions



Complex eigenvalues with Reλ<0:
decaying magnetic field

Complex eigenvalues with Reλ>0:
Oscillatory instability,
generation of the magnetic field,
oscillatory dynamo

in its kinematic regime (see, e.g., U. Guenther, F. Stefani, 2003)

Induction equation: $$\partial_t \mathbf{B} = \nabla \mathbf{B} \times (\alpha \mathbf{B}) + \nu_m \Delta \mathbf{B}, \quad \nabla \cdot \mathbf{B} = 0$$

Magnetic diffusivity: $\nu_m = const$

Helical turbulence function: $\alpha = \alpha(r)$

Decomposition into poloidal and toroidal components:

$$\mathbf{B} = \mathbf{B}_p + \mathbf{B}_t, \mathbf{B}_p = \nabla \times \mathbf{A}_t; \ \mathbf{A}_t = -\mathbf{r} \times \nabla F_1, \ \mathbf{B}_t = -\mathbf{r} \times \nabla F_2, \ \int_{S^2} F_{1,2} d\omega = 0$$

System of equations: $$r \times \nabla \left[\nu_m \Delta F_1 + \alpha F_2 - \partial_t F_1 \right] = 0$$

$$r \times \nabla \left[\nu_m \Delta F_2 - \frac{1}{r} (\partial_r \alpha)(\partial_r r F_1) - \alpha \Delta F_1 - \partial_t F_2 \right] = 0$$

Derivation of the α^2 – dynamo operator matrix

Re-scaling r and t: $\nu_{\mathrm{m}} = 1,$ boundary conditions at $r = 1$

A series expansion in spherical harmonics

$$F_{1,2} = \sum_{l,m,n} e^{t\lambda_{l,n}} F_{1,2}^{(l,m,n)}(r) Y_l^m(\theta,\phi) \in L^2(\Omega, r^2 dr) \otimes L^2(S^2, d\omega), \quad \Omega = [0,1]$$

Coupled l – modes of the poloidal and toroidal magnetic field components

$$\Delta_l F_1^{(l,m,n)} + \alpha F_2^{(l,m,n)} = \lambda_{l,n} F_1^{(l,m,n)}$$

$$\Delta_l F_2^{(l,m,n)} - \frac{1}{r}(\partial_r \alpha)(\partial_r r F_1^{(l.m.n)}) - \alpha \Delta_l F_1^{(l,m,n)} = \lambda_{l,n} F_2^{(l,m,n)}$$

$$\Delta_l := \frac{1}{r^2}\partial_r r^2 \partial_r - \frac{l(l+1)}{r^2}$$

Boundary eigenvalue problem for the operator matrix

$$\mathbf{L}(\lambda,\alpha)\mathbf{u} := \mathbf{l}_0\partial_x^2\mathbf{u} + \mathbf{l}_1\partial_x\mathbf{u} + \mathbf{l}_2\mathbf{u} = 0, \quad \mathbf{U}(\beta)\tilde{\mathbf{u}} := [\mathbf{A},\mathbf{B}]\tilde{\mathbf{u}} = 0, \quad x \in [0,1]$$

$$x = r, \quad \mathbf{u} := [u_1, u_2]^T = [F_1, F_2]^T, \quad \tilde{\mathbf{u}}^T := [\mathbf{u}^T(0), \partial_x\mathbf{u}^T(0), \mathbf{u}^T(1), \partial_x\mathbf{u}^T(1)]$$

$$\mathbf{l}_0 = \begin{pmatrix} 1 & 0 \\ \alpha(x) & 1 \end{pmatrix}, \quad \mathbf{l}_1 = \begin{pmatrix} 0 & 0 \\ -\partial_x\alpha(x) & 0 \end{pmatrix}, \quad \mathbf{l}_2 = \begin{pmatrix} -l(l+1)x^{-2} - \lambda & \alpha(x) \\ l(l+1)x^{-2}\alpha(x) & -l(l+1)x^{-2} - \lambda \end{pmatrix}$$

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \beta l + 1 - \beta & 0 & \beta & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Idealistic and physically realistic boundary conditions

Real eigenvalues $\lambda(\alpha_0)$ for $\alpha(x) = \alpha_0 = const$

Idealistic b. c.: $\beta = 0$

Zeros of Bessel functions:

$$J_{l+1/2}\left(\sqrt{\rho_n}\right) = 0, \quad 0 < \sqrt{\rho_1} < \sqrt{\rho_2} < \cdots$$

Spectral mesh, lines:

$$\lambda_n^{\pm}(\alpha_0) = -\rho_n \pm \alpha_0 \sqrt{\rho_n}, \quad n \in Z^+$$

l=0 :

$$\lambda_n^{\pm}(\alpha_0) = -(\pi n)^2 \pm \alpha_0 \pi n$$

Physically realistic b. c.: $\beta = 1$

No spectral mesh,

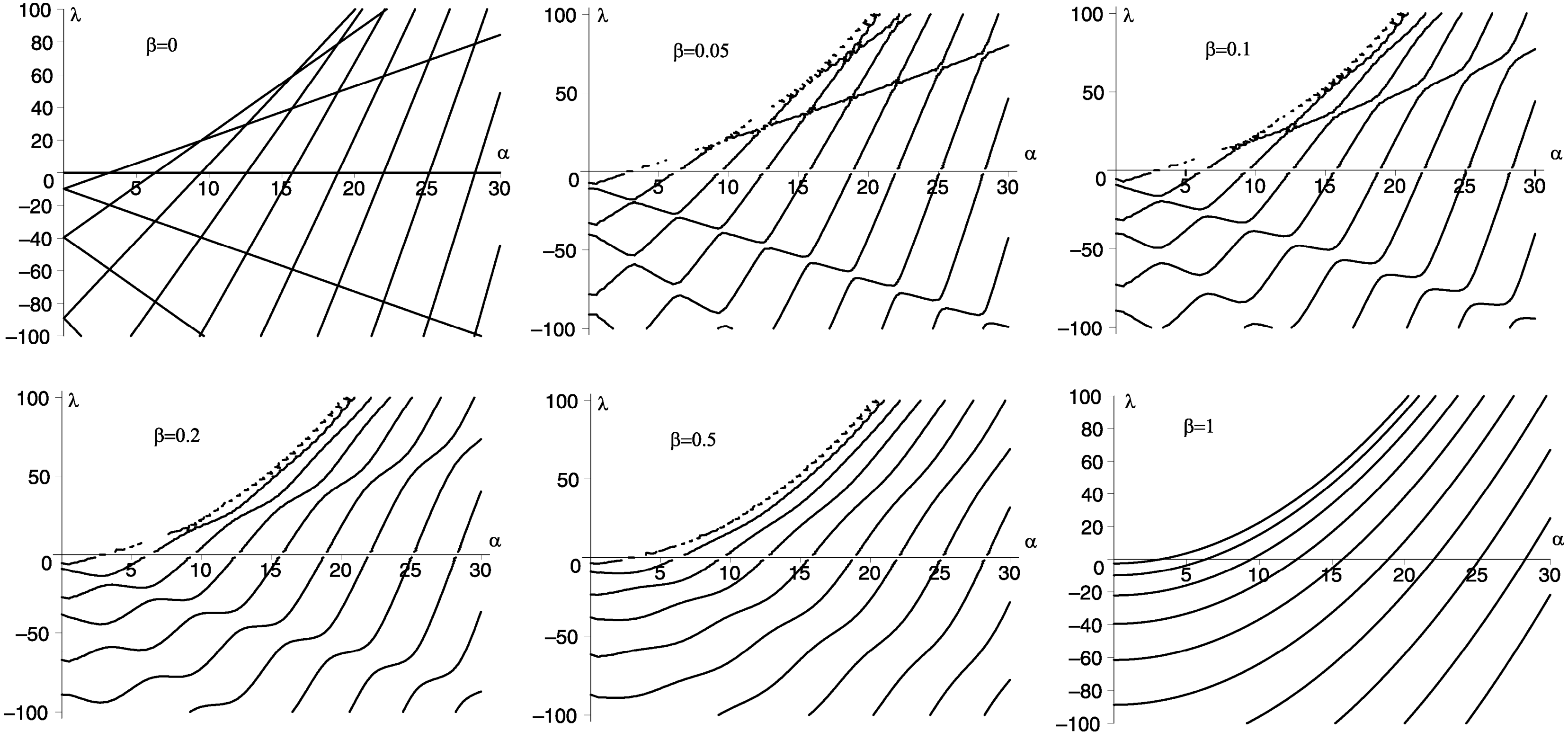
non-intersecting curves

(*l=0* – parabolas):

$$\lambda_n(\alpha_0) = \frac{1}{4}(\alpha_0^2 - \pi^2 n^2), \quad n \in N$$

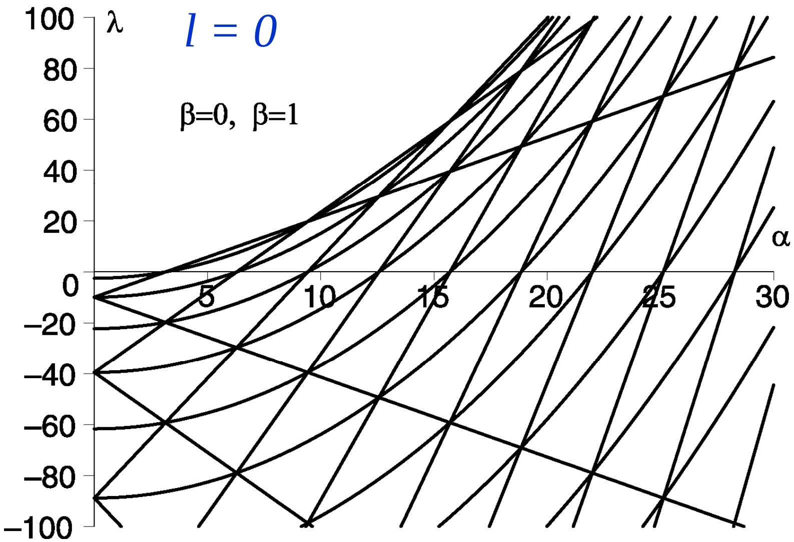
Deformation of the spectral mesh for $\alpha(x) = \alpha_0$

Homotopy parameter in the boundary conditions $\beta = 0 \ldots 1$

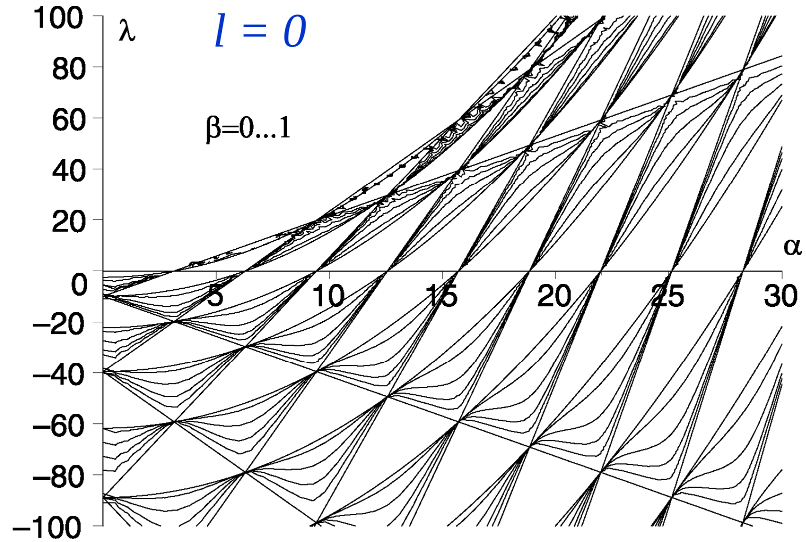


Transition from idealistic to physically realistic b. c.

$\beta = 0$ - lines, spectral mesh



$\beta = 1$ - parabolas, no mesh

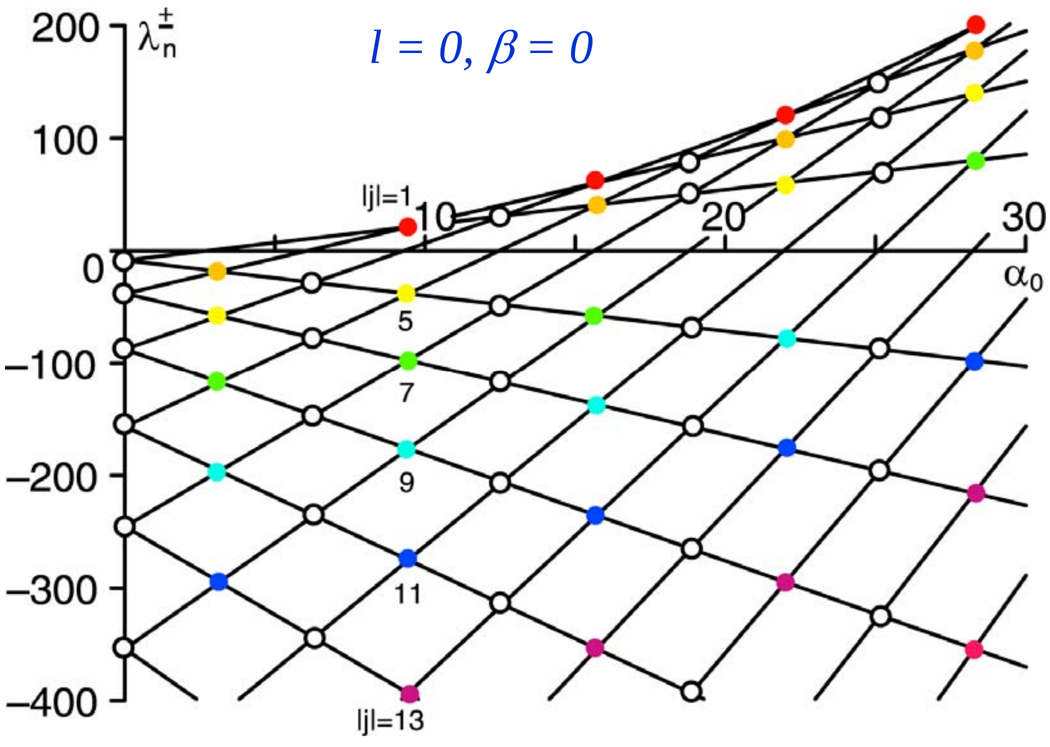


Nodes of the mesh for $\beta = 0$ belong to the parabolic curves described by the same equation as the spectrum for $\beta = 1$:

$$\lambda_n(\alpha_0) = (\alpha_0^2 - \pi^2 n^2)/4, \quad n \in Z$$

Diabolical points (DPs) at the nodes of the spectral mesh

Node: $\lambda_n^\varepsilon = \lambda_m^\delta, \quad \lambda_n^\varepsilon = -\rho_n + \varepsilon\alpha_0\sqrt{\rho_n}, \quad \varepsilon,\delta = \pm, \quad \mathbf{u}_n^\varepsilon = \begin{pmatrix} 1 \\ \varepsilon\sqrt{\rho_n} \end{pmatrix} u_n$



Nodal point: $(\alpha_0^\nu, \lambda_0^\nu)$

$$\alpha_0^\nu = \varepsilon\sqrt{\rho_n} + \delta\sqrt{\rho_m}$$

Double eigenvalue:

$$\lambda_0^\nu = \varepsilon\delta\sqrt{\rho_n\rho_m}$$

Two eigenvectors:

$$\mathbf{u}_n^\varepsilon, \quad \mathbf{u}_m^\delta$$

Unfolding DPs by perturbation $\alpha(x)=\alpha_0^\nu+\Delta\alpha(x),\ \beta=0$

Indefinite inner product in Krein space: $[.,.]=(\mathbf{J}.,.)$

$$\mathbf{J}=\begin{pmatrix}0 & I\\ I & 0\end{pmatrix}$$

Krein space state of positive type ($[\mathbf{u}_n^+,\mathbf{u}_n^+]>0$)
= positive slope of the eigenvalue branch $\lambda_n^+(\alpha_0)$

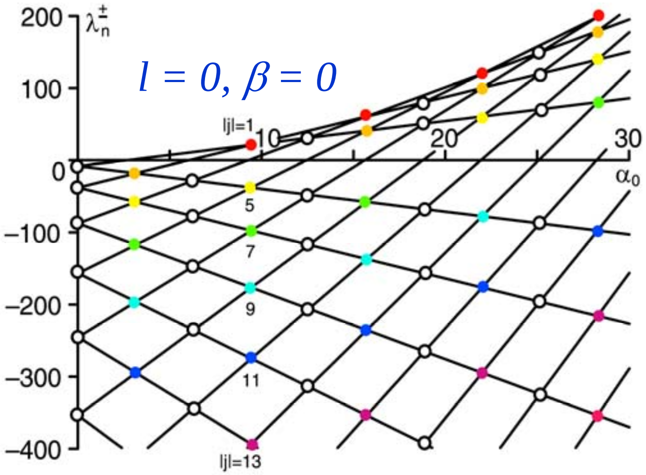
Define: $$a_1=\varepsilon\frac{[\mathbf{Ku}_n^\varepsilon,\mathbf{u}_n^\varepsilon]}{\sqrt{\rho_n}},\quad a_2=\delta\frac{[\mathbf{Ku}_m^\delta,\mathbf{u}_m^\delta]}{\sqrt{\rho_m}},\quad b^2=\frac{[\mathbf{Ku}_n^\varepsilon,\mathbf{u}_n^\varepsilon][\mathbf{Ku}_m^\delta,\mathbf{u}_m^\delta]}{\sqrt{\rho_n\rho_m}}$$

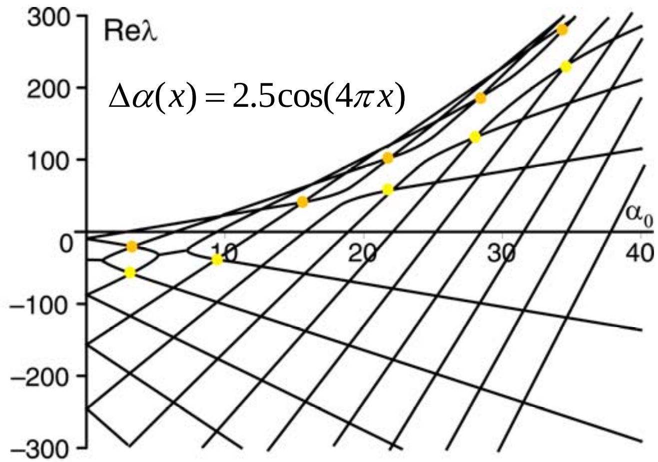
$$[\mathbf{Ku}_n^\varepsilon,\mathbf{u}_m^\delta]=\int_0^1\Delta\alpha\left[\left(\varepsilon\delta\sqrt{\rho_n\rho_m}+\frac{l(l+1)}{x^2}\right)u_m u_n+u_m' u_n'\right]dx$$

Splitting: $\lambda=\lambda_0^\nu+\Delta\lambda,\quad \Delta\lambda=\left[(a_1+a_2)\pm\sqrt{(a_1-a_2)^2+\varepsilon\delta\, b^2}\,\right]/2$

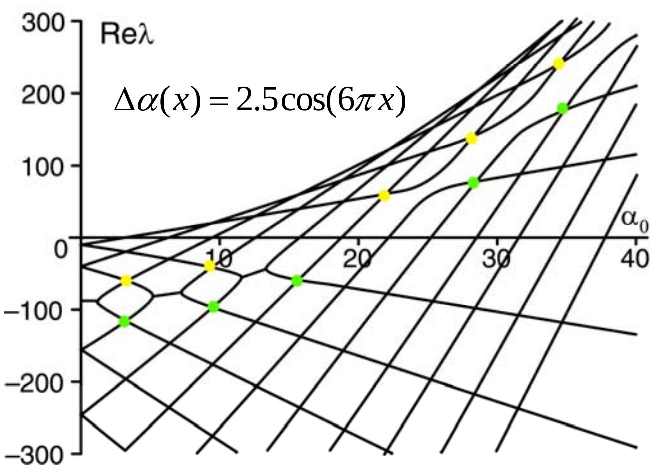
$\varepsilon=\delta$ ($\lambda_0^\nu>0$): $\Delta\lambda$ is real; $\varepsilon=-\delta$ ($\lambda_0^\nu<0$): $\Delta\lambda$ can be complex

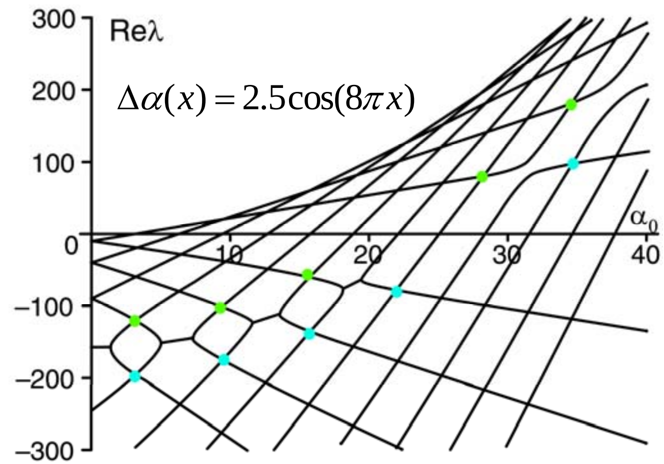
Two branches with the slopes of the same sign – real splitting





Two branches with the slopes of different signs – complex splitting





Resonance effect

Diabolical point at $(n, n+j)$ node of the spectral mesh:

$$l = 0: \quad \lambda_0^\nu = \pi^2 n(n+j), \quad \alpha_0^\nu = \pi(2n+j), \quad n, j \in Z^*$$

Fourier coefficients of the perturbation $\Delta\alpha(x)$:

$$a_0 = 2\int_0^1 \Delta\alpha(x)dx, \quad a_k = 2\int_0^1 \Delta\alpha(x)\cos(2\pi kx)dx, \quad b_k = 2\int_0^1 \Delta\alpha(x)\sin(2\pi kx)dx$$

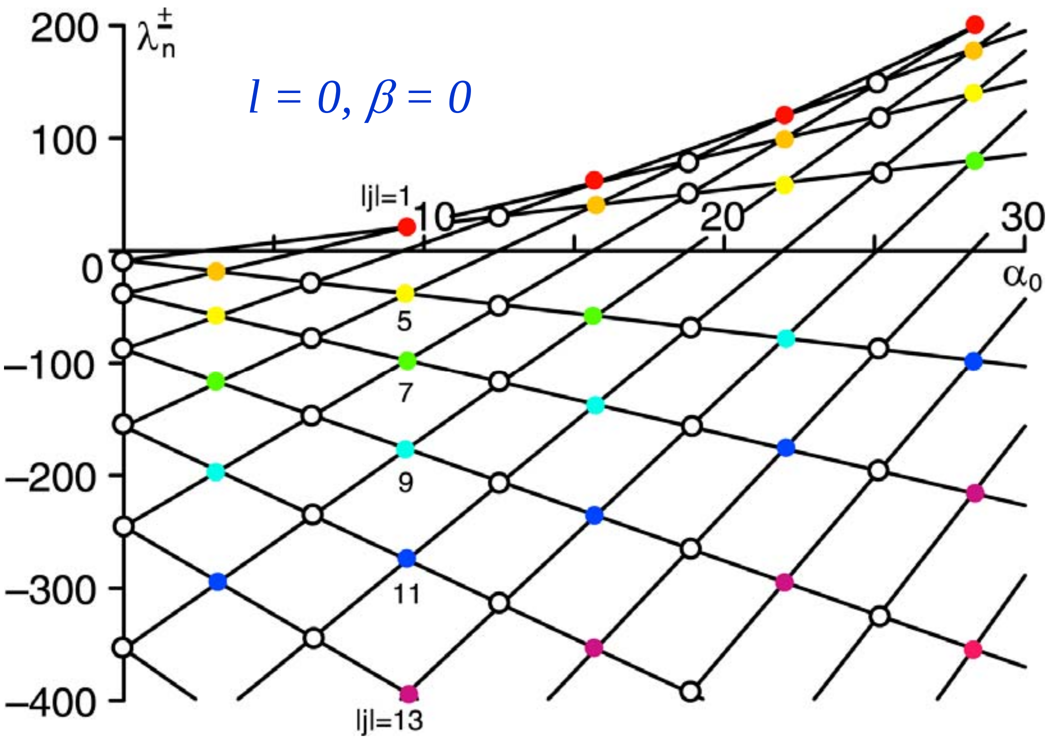
Perturbation of the eigenvalue at the node:

$$\lambda = \lambda_0^\nu + \frac{\pi}{4}[(2n+j)a_0 \pm \sqrt{j^2 a_0^2 + 4n(n+j)Q_j^2}\,]$$

Resonance term: $$Q_j = \begin{cases} \dfrac{8}{\pi}\sum_{k=1}^{\infty} b_k \dfrac{k}{4k^2 - j^2}, & j = \pm 1, \pm 3, \dots \\ a_k(\delta_{j,2k} + \delta_{-j,2k}), & j = \pm 2, \pm 4, \dots \end{cases}$$

Resonance pattern of the spectral mesh

The diabolical points at the nodes $(n, n+j)$ with the same $|\,j\,|$ are located on a parabolic curve: $\lambda_0^\nu = \frac{1}{4}(\alpha_0^{\nu\,2} - \pi^2 j^2)$



Colored points: odd j

White points: even j

Preliminary conclusions

- $\alpha(x) = \alpha_0 = const, \quad \beta \in [0,1]$: All the eigenvalues are real. Oscillatory instability (dynamo) does not exist

- $\alpha(x) = \alpha_0 + \Delta\alpha(x), \quad \beta = 0$: In the first approximation only the nodes selected by the Fourier components of $\Delta\alpha(x)$ split into distinct eigenvalues

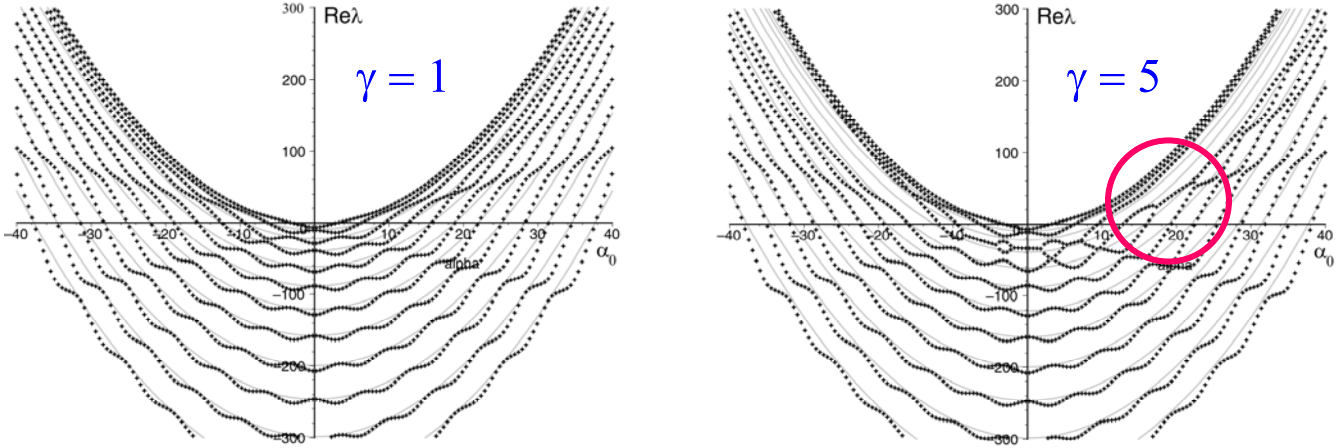
- The nodes with $\lambda_0^\nu > 0$ originate positive real eigenvalues in the avoided crossings (non-oscillatory dynamo regime)

- The nodes with $\lambda_0^\nu < 0$ can produce complex eigenvalues with $\mathrm{Re}\,\lambda < 0$ (decaying magnetic field)

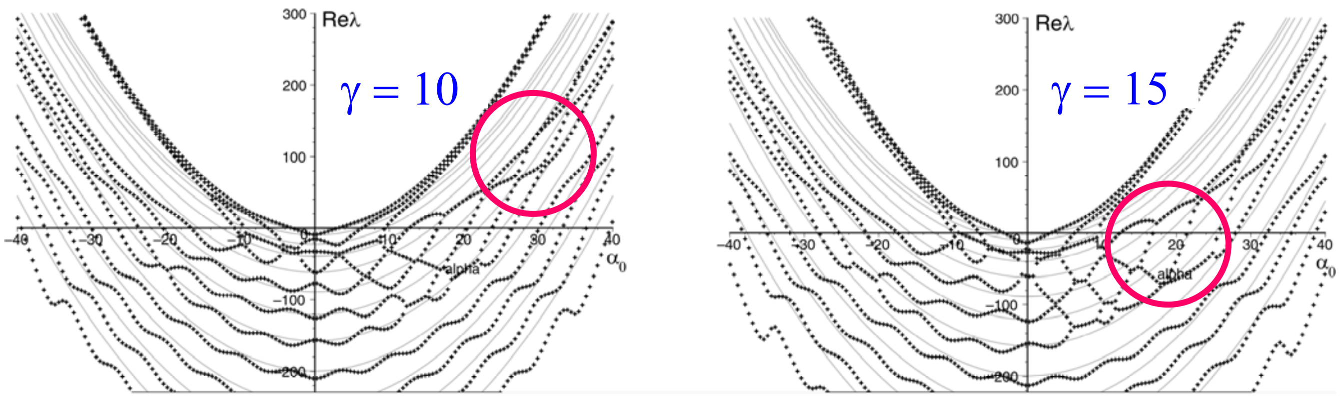
- How to get complex eigenvalues with $\mathrm{Re}\,\lambda > 0$?

Simultaneous perturbation of the α - profile and b. c.

Given $\varphi(x)$: $\alpha(x) = \alpha_0 + \gamma\,\varphi(x)$; 3 parameters: α_0, β, and γ



Complex eigenvalues with Rel > 0 in the „prohibited region“

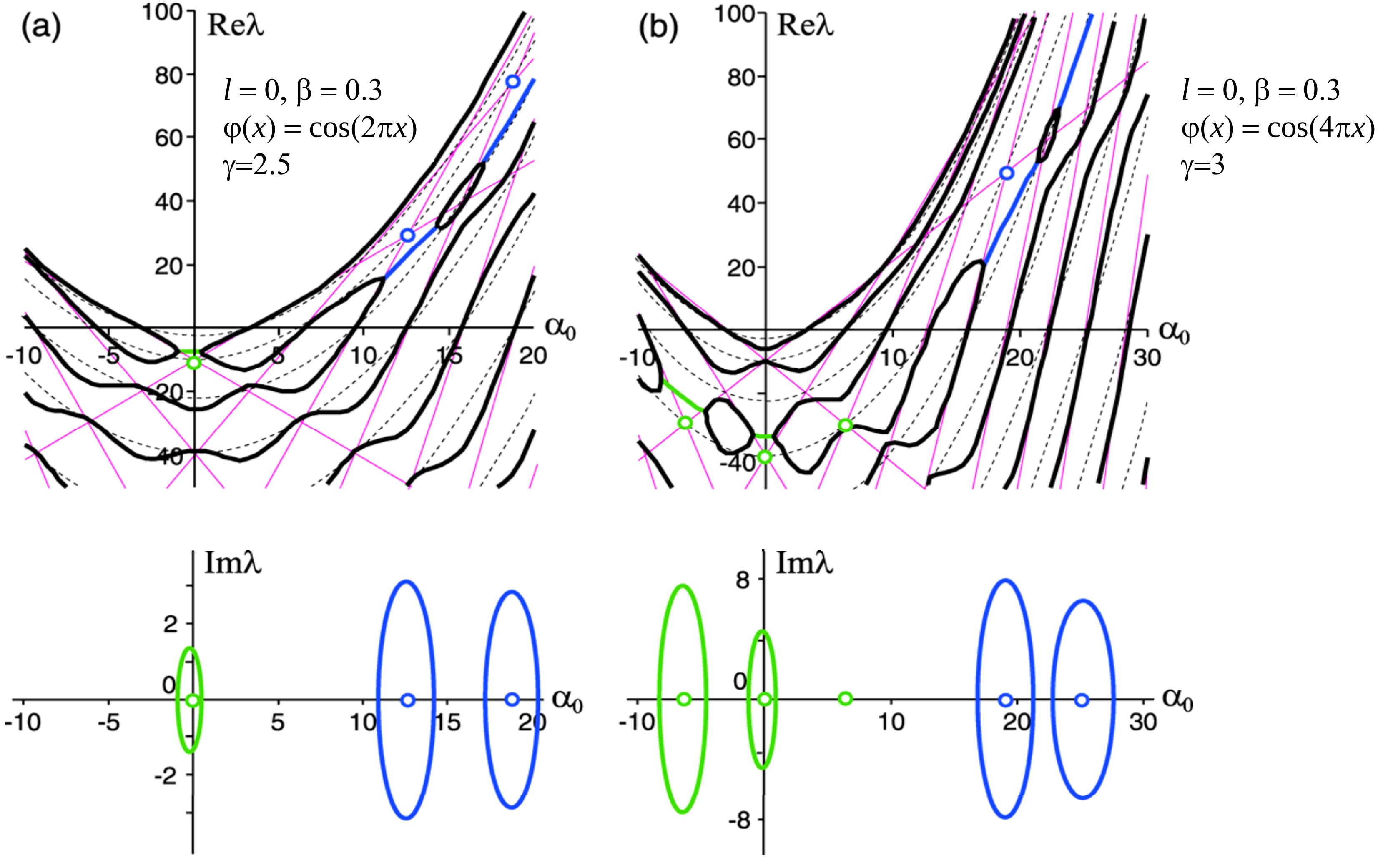


$l = 0$

$\varphi(x) = \cos(4\pi x)$

$\beta = 0.2$

Resonant influence of the ‘hidden’ diabolical points



α, β, γ – splitting the diabolical points

Node: $(\alpha_0^\nu, \lambda_0^\nu), \quad \lambda_0^\nu = \varepsilon\delta\pi^2 nm, \quad \alpha_0^\nu = \varepsilon\pi n + \delta\pi m, \quad \varepsilon, \delta = \pm$

After the splitting[1]:

$$\lambda(\alpha_0, \beta, \gamma) = \lambda_0^\nu - \varepsilon\delta\pi^2 nm\beta + \frac{\pi}{2}(\delta m + \varepsilon n)\Delta\alpha_0$$
$$\pm\frac{\pi}{2}\sqrt{((\delta m - \varepsilon n)\Delta\alpha_0)^2 + 4mn\left(\varepsilon\gamma\Delta\alpha - (-1)^{n+m}\pi n\beta\right)\left(\delta\gamma\Delta\alpha - (-1)^{n+m}\pi m\beta\right)}$$

Conditions for existence of the complex eigenvalues:

$$((\varepsilon n - \delta m)\Delta\alpha_0)^2 +$$
$$mn((\varepsilon + \delta)\gamma\Delta\alpha - (-1)^{n+m}(n+m)\beta\pi)^2 - mn((\varepsilon - \delta)\gamma\Delta\alpha - (-1)^{n+m}(n-m)\beta\pi)^2 < 0$$

1: $\Delta\alpha_0 := \alpha_0 - \alpha_0^\nu, \quad \Delta\alpha := \int_0^1 \varphi(x)\cos((\varepsilon n - \delta m)\pi x)dx, \quad \int_0^1 \varphi(x)dx = 0$

Approximation of the resonance tongues

$$\varphi(x) = \cos(2\pi k x), \quad \Delta\alpha = \int_0^1 \varphi(x)\cos((\varepsilon n - \delta m)x)dx = \begin{cases} 1/2, & 2k = \pm(\varepsilon n - \delta m) \\ 0, & 2k \neq \pm(\varepsilon n - \delta m) \end{cases}$$

Two sorts of the resonance domains in the plane (α_0, γ)

Hyperbolic regions for $\varepsilon = -\delta$ (decaying magnetic field):

$$mn(\varepsilon\gamma - (n-m)\beta\pi)^2 - 4k^2(\alpha_0 - \alpha_0^\nu)^2 > mn((n+m)\beta\pi)^2$$

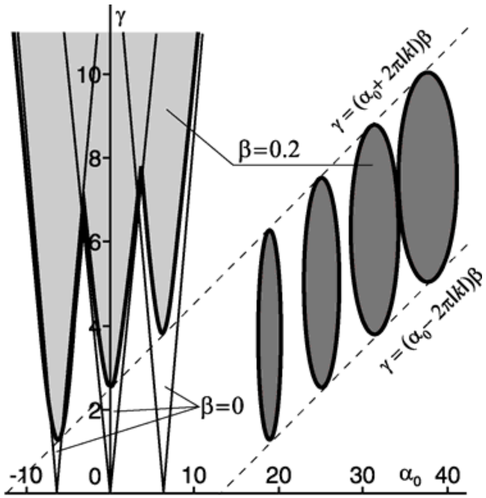
$$\mathrm{Re}\,\lambda = -\pi^2 mn(1-\beta) + \varepsilon\frac{\pi}{2}(n-m)(\alpha_0 - \alpha_0^\nu) < 0$$

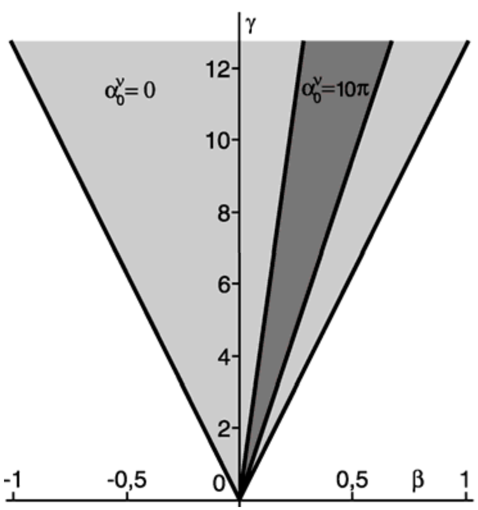
Elliptic regions for $\varepsilon = \delta$ (growing magnetic field):

$$4k^2(\alpha_0 - \alpha_0^\nu)^2 + mn(\varepsilon\gamma - (n+m)\beta\pi)^2 < mn((n-m)\beta\pi)^2$$

$$\mathrm{Re}\,\lambda = \pi^2 mn(1-\beta) + \varepsilon\frac{\pi}{2}(n+m)(\alpha_0 - \alpha_0^\nu) > 0$$

Resonance tongues and balloons ($\varphi(x)=\cos(2\pi kx)$, $k = 2$)





- $\beta = 0$: only resonant tongues with Re $\lambda < 0$ are visible in the plane (α_0, γ)
- $\beta \neq 0$: resonance “balloons” of oscillatory dynamo with Re $\lambda > 0$ appear in the “prohibited” zone in the plane (α_0, γ)
- Separate variation of γ or β does not yield the oscillatory dynamo
- Simultaneous changing of γ and β easily produces the non-trivial oscillatory dynamo regions
- γ is bounded from the above (in a corridor)

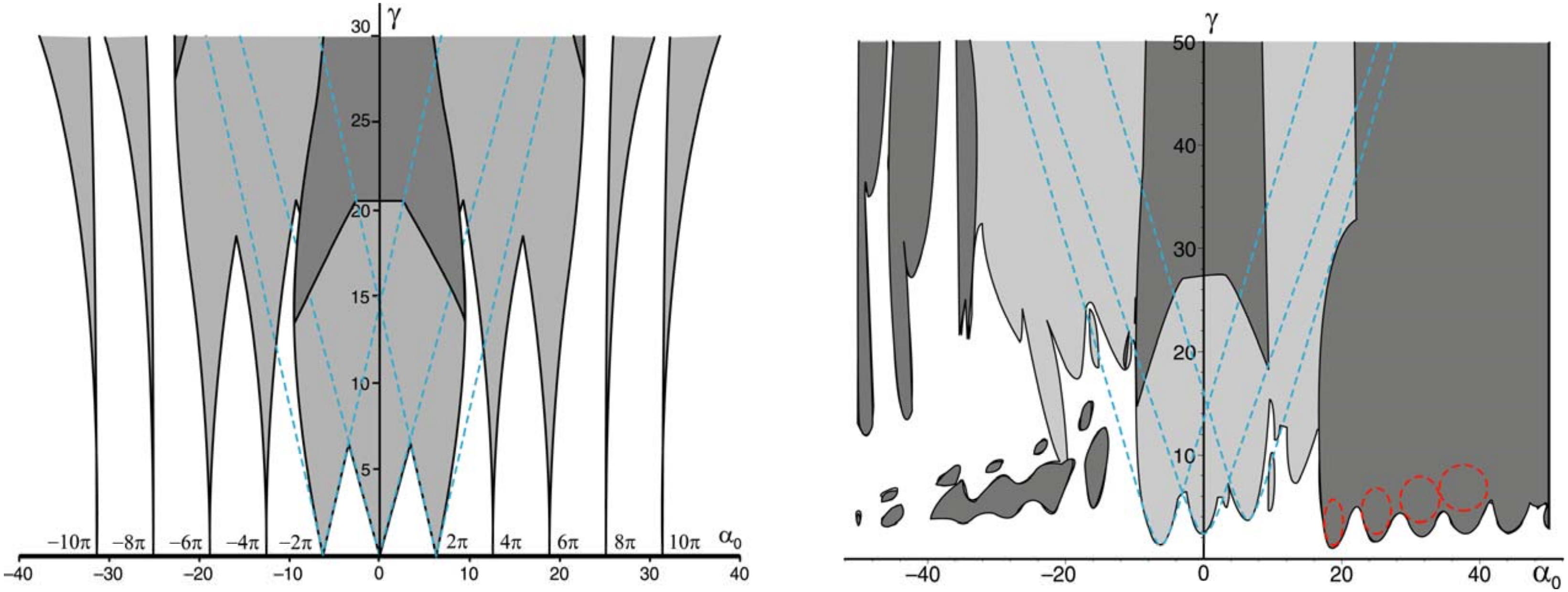
Comparison with the numerical results ($\varphi(x)$=cos($2\pi kx$), $k = 2$)

3 principal tongues:

$$\gamma^2 - 4\alpha_0^2 > 16\pi^2\beta^2, \quad 16(\alpha_0 \pm 2\pi)^2 + (\gamma \pm 10\pi\beta)^2 < 4(\gamma \pm 4\pi\beta)^2$$

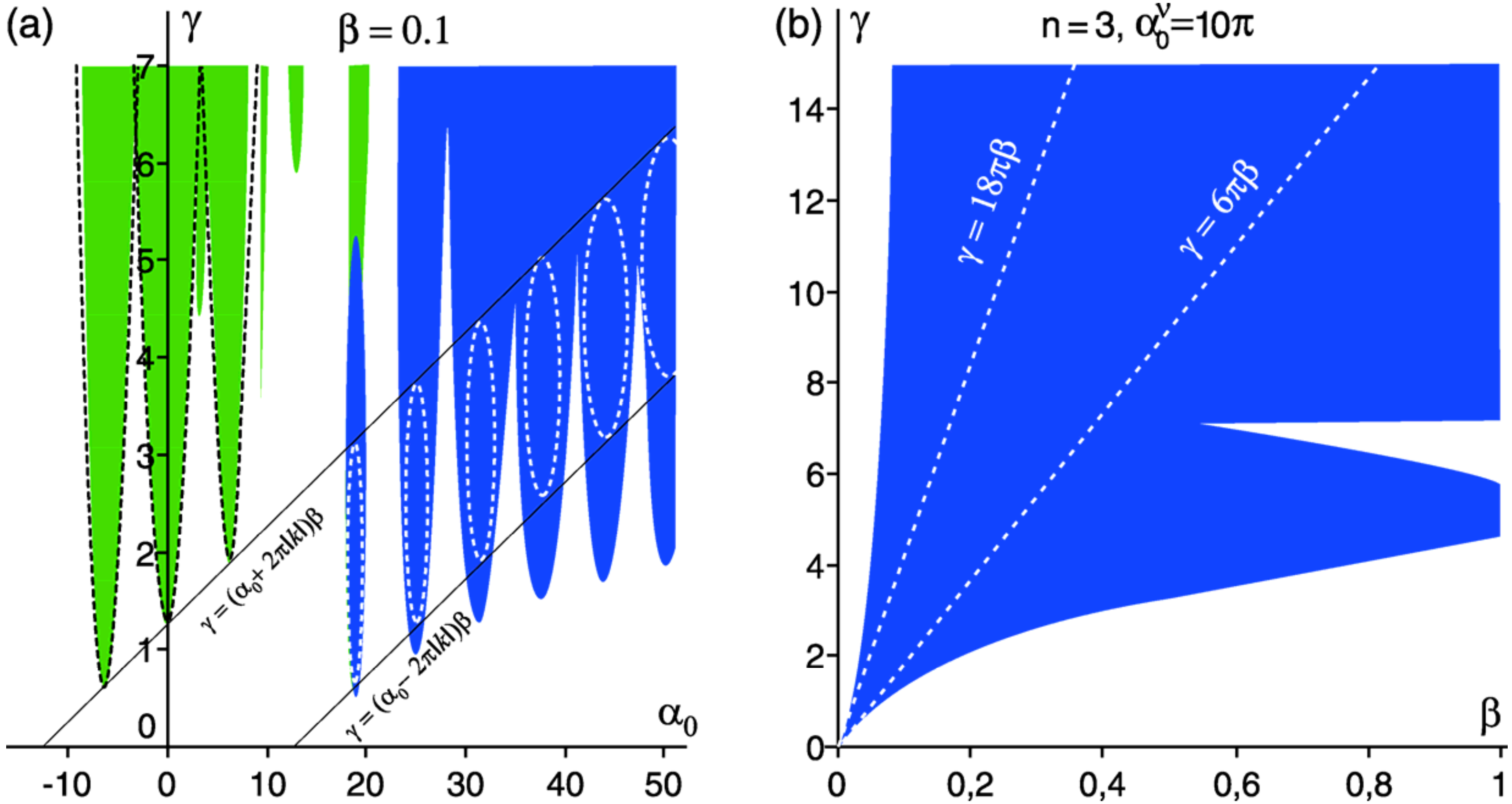
Infinitely many balloons: n = 1,2,…

$$16(\alpha_0 - (4+2n)\pi)^2 + n(4+n)(\gamma + 2(n+2)\beta\pi)^2 < 4n(4+n)(k\beta\pi)^2$$



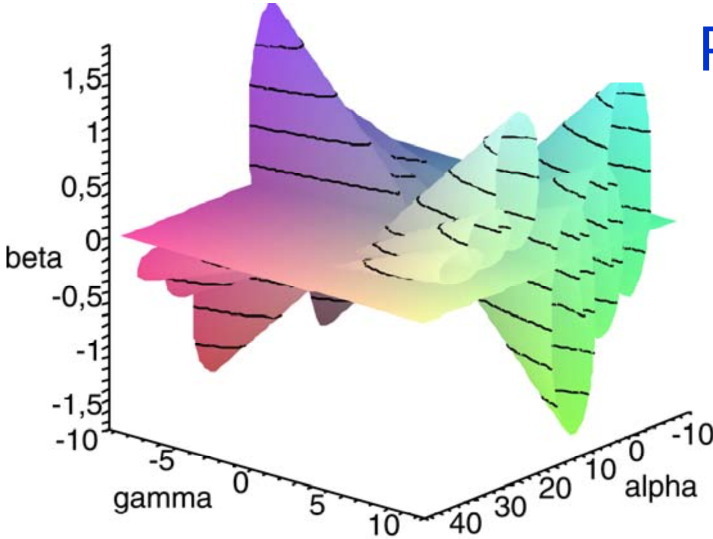
$\beta = 0$ $\beta = 0.2$

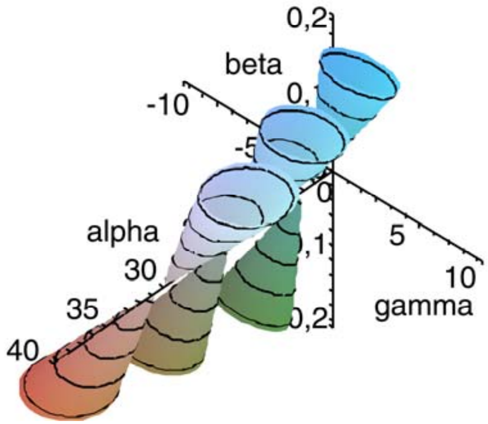
Comparison with the numerical results

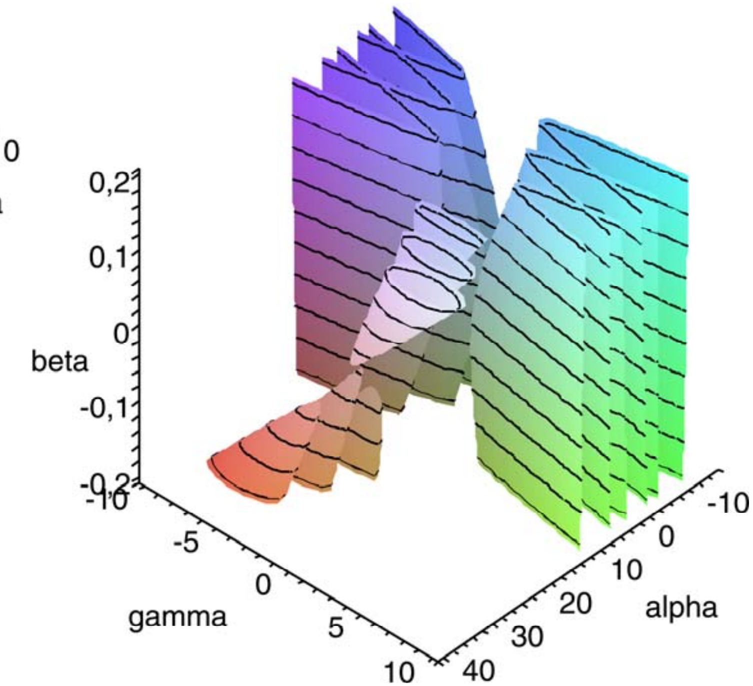


3d pictures of Arnold’s tongues ($\varphi(x) = \cos(2\pi kx)$, $k = 2$)

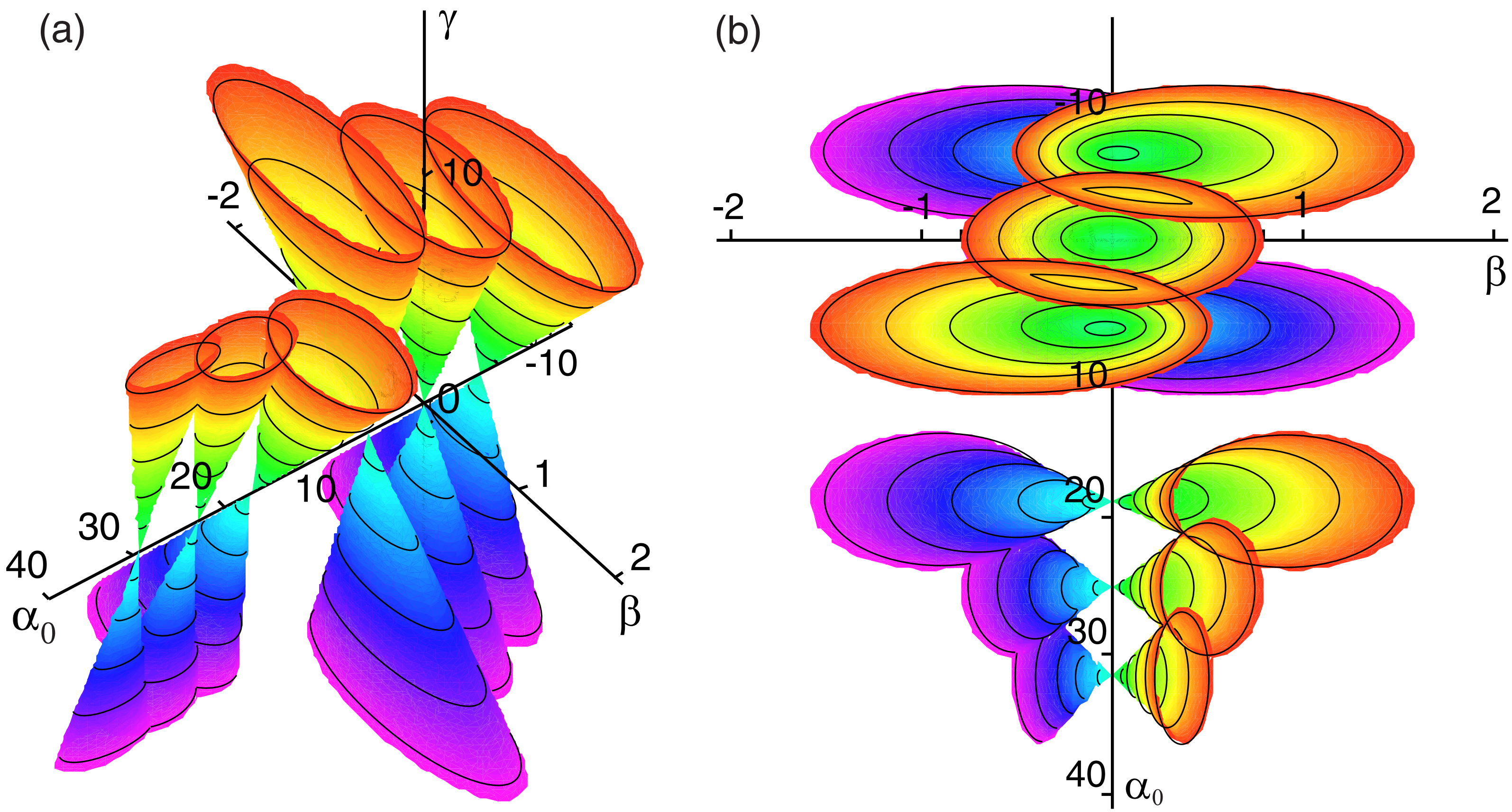
Planar tongues and balloons are projections of differently inclined 3d cones







Two groups of Arnold's tongues selected by the Krein signature at DPs



Conclusions

It looks non-trivial to get the oscillatory instability in the regions were the Krein signature of the crossed modes is the same

Nevertheless, the answer is simple:

Oscillatory instability is inside the conical tongues which start at the diabolical points for idealistic boundary conditions

Just change the point of view…

Thank you!

http://www.dyn.tu-darmstadt.de/members/kirillov/

http://onkirillov.narod.ru