Chapter 4

Reversible and circulatory systems

In the case of complicated structures there may appear different shapes of characteristic curves, and only an analysis in the [load-frequency] plane may assure the correct results for the design of structures subjected to nonconservative loads.

O. Mahrenholtz, R. Bogacz [396]

4.1 Reversible systems

Equations of *reversible* dynamical systems are invariant under a particular type of coordinate transformation which is accompanied, e.g, by *time reversal* [344, 459]. We define the dynamical system

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}), \tag{4.1}$$

where $\mathbf{x}$, $\mathbf{f} \in \mathbb{R}^q$, to be reversible if there exists an involution of the state variables $\mathbf{R} = \mathbf{R}^{-1}$ such that [460, 575]

$$\frac{d\mathbf{x}}{dt} = -\mathbf{R}\mathbf{f}(\mathbf{R}\mathbf{x}) = \mathbf{f}(\mathbf{x}), \tag{4.2}$$

leaving equation (4.1) invariant under the transformation[1] $\mathbf{x} \to \mathbf{R}\mathbf{x}$ and $t \to -t$.

In particular, all oscillation equations of the following form are reversible

$$\frac{d^2\mathbf{z}}{dt^2} = \mathbf{g}(\mathbf{z}), \tag{4.3}$$

where $\mathbf{z}$, $\mathbf{g} \in \mathbb{R}^m$. When (4.3) is rewritten as a first order system in the variables z_s and dz_s/dt in $\mathbb{R}^q$, $q = 2m$, the involution changes the signs of dz_s/dt [344].

From reversibility, at an equilibrium $\mathbf{x}_0$ of equation (4.1) the Jacobian

$$\mathbf{A} := D\mathbf{f}(\mathbf{x}_0) = -\mathbf{R}\mathbf{A}\mathbf{R}, \tag{4.4}$$

[1] In quantum mechanics, Wigner's time reversal operation [158, 595] leaves the 'Schrödinger equation' $\frac{d\mathbf{x}}{dt} = i\mathbf{A}\mathbf{x}$ invariant under the transformation $\mathbf{x} \to \mathbf{U}\overline{\mathbf{x}}$ and $t \to -t$, where $i = \sqrt{-1}$, the overbar denotes complex conjugation, and $\mathbf{U}$ is a unitary matrix, $\mathbf{U}\overline{\mathbf{U}}^T = \overline{\mathbf{U}}^T\mathbf{U} = \mathbf{I}$. Indeed, the *anti-unitary symmetry*, $\mathbf{A} = \mathbf{U}\,\overline{\mathbf{A}}\,\overline{\mathbf{U}}^T$, transforms the equation $\frac{d\mathbf{U}\overline{\mathbf{x}}}{dt} = -i\mathbf{A}\mathbf{U}\overline{\mathbf{x}}$, into $\frac{d\mathbf{x}}{dt} = i\mathbf{U}^T\,\overline{\mathbf{A}}\,\overline{\mathbf{U}}\mathbf{x} = i\mathbf{A}\mathbf{x}$. The operator, $\mathbf{A}$, of the time reversible 'quantum' system $\frac{d\mathbf{x}}{dt} = i\mathbf{A}\mathbf{x}$, can be a real or complex symmetric matrix $\mathbf{A} = \mathbf{A}^T$ [212, 595].