Doklady Physics, Vol. 46, No. 3, 2001, pp. 184–189. From Doklady Akademii Nauk, Vol. 377, No. 1, 2001, pp. 44–49.
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MECHANICS

Overlapping of Frequency Curves in Nonconservative Systems[1]

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Presented by Academician S.S. Grigoryan July 14, 2000

Received July 18, 2000

The overlapping of characteristic curves often arises in nonconservative systems, depending on the parameters involved [1–4]. The characteristic curves describe the eigenvalues of an operator for a system as functions of a chosen parameter, for example, a parameter of a nonconservative load. The phenomenon consists in the fact that, when varying parameters, the characteristic curves come closer together, merge at a point, and then overlap, forming a “bubble of instability”. In this case, the overlapping is accompanied by a discontinuity of the critical load.

In this paper, explicit formulas describing the overlapping in two-parametric nonconservative systems are derived. These formulas use information on a system only at the merging point and allow us to perform qualitative, as well as quantitative, analyses for the behavior of the characteristic curves in the vicinity of this point. A quadratic approximation of the flutter boundary is also obtained. It is shown that the features of the characteristic curves are related to convexity properties of the flutter instability boundary.

1. We consider a linear autonomous nonconservative mechanical system without damping and gyroscopic forces, which is described by the equation

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\mathbf{q} = \mathbf{0}. \tag{1}$$

Here, $\mathbf{M} = \mathbf{M}^{\mathrm{T}} > 0$ and $\mathbf{C} \neq \mathbf{C}^{\mathrm{T}}$ are real $m \times m$ matrices of mass and stiffness, $\mathbf{q}$ is the m-dimensional vector of generalized coordinates, and the dots stand for the differentiation with respect to time. System (1) is usually referred to as a circulatory system [5, 6]. Let ω be the frequency of oscillations. Substituting $\mathbf{q} = \mathbf{u}\exp(i\omega t)$ into (1) and introducing the notation $\mathbf{A} = \mathbf{M}^{-1}\mathbf{C}$ and $\lambda = \omega^2$, we arrive at an eigenvalue problem:

$$\mathbf{A}\mathbf{u} = \lambda\mathbf{u}. \tag{2}$$

System (1) is stable if all the eigenvalues of problem (2) are positive and semisimple. If all the eigenvalues λ are real and some of them negative, then system (1) is statically unstable (the case of divergence). If at least one eigenvalue λ is complex-valued, vibrational instability (i.e., flutter) occurs.

2. We suppose that the matrix $\mathbf{A}$ smoothly depends on the two-dimensional vector $\mathbf{p} = (p_1, p_2)$ of real-valued parameters. It is known [7–9] that, in general, the stability boundary of a two-parametric circulatory system consists of smooth curves on which the matrix $\mathbf{A}$ has either a simple zero eigenvalue or a positive double eigenvalue with a Jordan chain of the length of 2. At individual points of the stability boundary, singularities of two types (namely, cusps and nodes) are possible. These singularities correspond to matrices with a more complicated Jordan structure [7, 9].

We first consider a point $\mathbf{p} = \mathbf{p}_0$ on the stability boundary on which the matrix $\mathbf{A}_0 = \mathbf{A}(\mathbf{p}_0)$ has a positive double eigenvalue λ_0 with a Jordan chain of the length of 2. At this point, the eigenvalue λ_0 corresponds to an eigenvector $\mathbf{u}_0$, associated vector $\mathbf{u}_1$, adjoint eigenvector $\mathbf{v}_0$, and adjoint associated vector $\mathbf{v}_1$, which are governed by the equations

$$(\mathbf{A}_0 - \lambda_0\mathbf{I})\mathbf{u}_0 = \mathbf{0}, \quad (\mathbf{A}_0 - \lambda_0\mathbf{I})\mathbf{u}_1 = \mathbf{u}_0, \tag{3}$$

$$(\mathbf{A}_0^{\mathrm{T}} - \lambda_0\mathbf{I})\mathbf{v}_0 = \mathbf{0}, \quad (\mathbf{A}_0^{\mathrm{T}} - \lambda_0\mathbf{I})\mathbf{v}_1 = \mathbf{v}_0, \tag{4}$$

where $\mathbf{I}$ is the identity matrix. The vectors $\mathbf{u}_0$, $\mathbf{u}_1$, $\mathbf{v}_0$, and $\mathbf{v}_1$ are related by conditions of orthogonality and normalization:

$$(\mathbf{u}_0, \mathbf{v}_0) = 0, \quad (\mathbf{u}_1, \mathbf{v}_0) \equiv (\mathbf{u}_0, \mathbf{v}_1) = 1. \tag{5}$$

The parentheses in (5) stand for the Hermitian scalar product $(\mathbf{a}, \mathbf{b}) = \sum_{i=1}^{m} a_i\bar{b}_i$ of the vectors $\mathbf{a}, \mathbf{b} \in C^m$.

We consider smooth variations $\mathbf{p}(\varepsilon) = \mathbf{p}_0 + \varepsilon\mathbf{e} + \varepsilon^2\mathbf{d}$ of the parametric vector in the vicinity of the point $\mathbf{p}_0$. Here, $\mathbf{e}$ and $\mathbf{d} \in R^2$ are variation vectors, $|\mathbf{e}| = 1$, and

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[1] This article was submitted by the authors in English.

$\varepsilon > 0$ is a small parameter. As a result of this perturbation, the matrix $\mathbf{A}_0$ takes the increment

$$\mathbf{A}(\mathbf{p}_0 + \varepsilon\mathbf{e} + \varepsilon^2\mathbf{d}) = \mathbf{A}_0 + \varepsilon\mathbf{A}_1 + \varepsilon^2\mathbf{A}_2 + \ldots, \quad (6)$$

where matrices $\mathbf{A}_1$ and $\mathbf{A}_2$ are given by

$$\mathbf{A}_1 = \sum_{s=1}^{2} \frac{\partial \mathbf{A}}{\partial p_s} e_s,$$
$$\mathbf{A}_2 = \sum_{i=1}^{2} \frac{\partial \mathbf{A}}{\partial p_i} d_i + \frac{1}{2} \sum_{s,t=1}^{2} \frac{\partial^2 \mathbf{A}}{\partial p_s \partial p_t} e_s e_t. \quad (7)$$

The derivatives in (7) are taken at the point $\mathbf{p} = \mathbf{p}_0$.

Due to the variation of the vector of parameters, both the eigenvalue λ_0 and the corresponding eigenvector $\mathbf{u}_0$ also take increments. According to the perturbation theory of non-self-adjoint operators [10], in the case of a double eigenvalue with a Jordan chain of the length of 2, the expansions for the eigenvalue and eigenvector contain terms with fractional powers of the small parameter ε:

$$\lambda = \lambda_0 + \varepsilon^{1/2}\lambda_1 + \varepsilon\lambda_2 + \varepsilon^{3/2}\lambda_3 + \ldots,$$
$$\mathbf{u} = \mathbf{u}_0 + \varepsilon^{1/2}\mathbf{w}_1 + \varepsilon\mathbf{w}_2 + \varepsilon^{3/2}\mathbf{w}_3 + \ldots \quad (8)$$

Substituting expansions (6) and (8) into eigenvalue problem (2), we obtain equations for determining the perturbations of λ_0 and $\mathbf{u}_0$. For the first coefficient λ_1, we have

$$\lambda_1 = \pm\sqrt{(\mathbf{A}_1\mathbf{u}_0, \mathbf{v}_0)}. \quad (9)$$

Using expression (7) for the matrix $\mathbf{A}_1$, we introduce a vector $\mathbf{f}$ with the components

$$f^s = \left(\frac{\partial \mathbf{A}}{\partial p_s}\mathbf{u}_0, \mathbf{v}_0\right), \quad s = 1, 2. \quad (10)$$

As λ_0 is a real eigenvalue, the vector $\mathbf{f}$ is also real. With regard to (10), we obtain a formula describing the splitting of the double eigenvalue [9]:

$$\lambda = \lambda_0 \pm \sqrt{\varepsilon\langle \mathbf{f}, \mathbf{e}\rangle} + O(\varepsilon). \quad (11)$$

Here, $\langle \mathbf{a}, \mathbf{b}\rangle = \sum_{s=1}^{2} a_s b_s$ is the scalar product of the vectors $\mathbf{a}, \mathbf{b} \in R^2$ in the parameter space. Formula (11) is valid if the radicand is not zero. If $\langle \mathbf{f}, \mathbf{e}\rangle < 0$, then the double eigenvalue λ_0 splits into a complex-conjugate pair of eigenvalues (flutter instability). If $\langle \mathbf{f}, \mathbf{e}\rangle > 0$, two positive eigenvalues appear (stability). Therefore, the vector $\mathbf{f}$ is a normal to the stability boundary and lies in the stability domain (see Fig. 1a).

We now consider a degenerate case when

$$\langle \mathbf{f}, \mathbf{e}_* \rangle = 0. \quad (12)$$

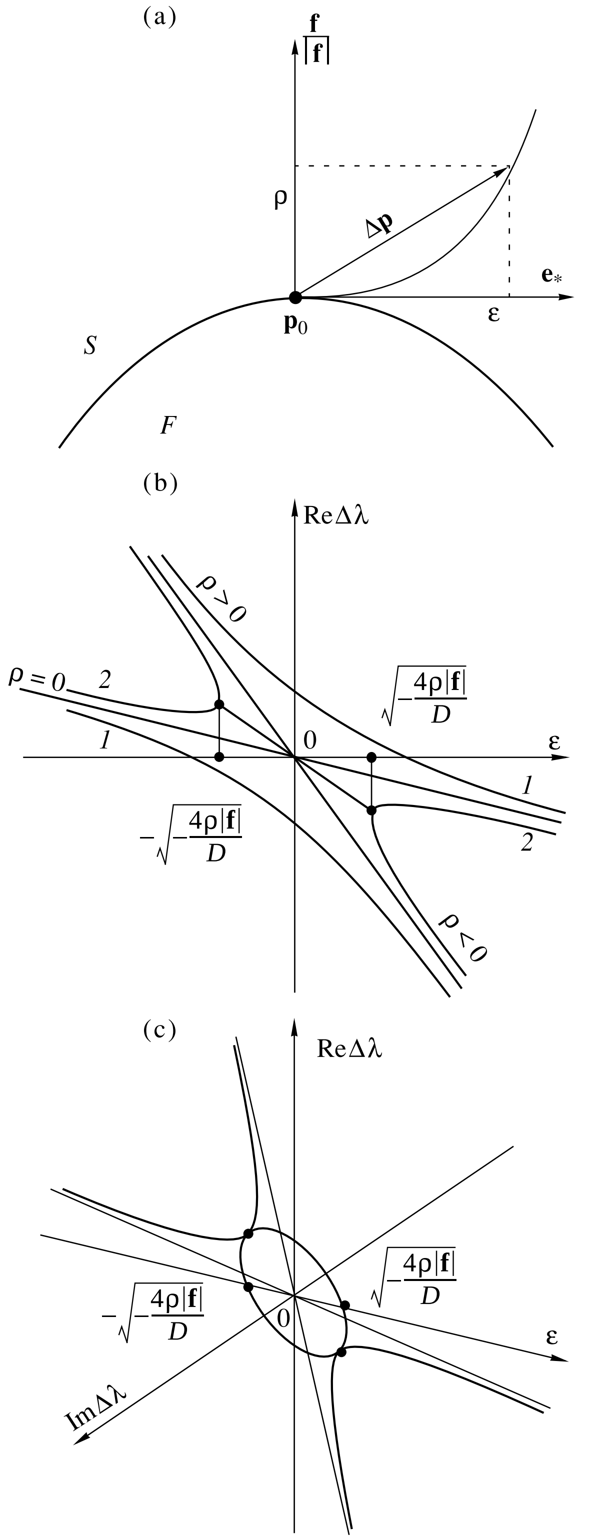

Fig. 1. Transformation of frequency curves near the flutter boundary.

This implies that the curves $\mathbf{p}(\varepsilon) = \mathbf{p}_0 + \varepsilon\mathbf{e}_* + \varepsilon^2\mathbf{d}$ tend to the tangents to the stability boundary as $\varepsilon \longrightarrow 0$. In this case, the terms with fractional powers disappear

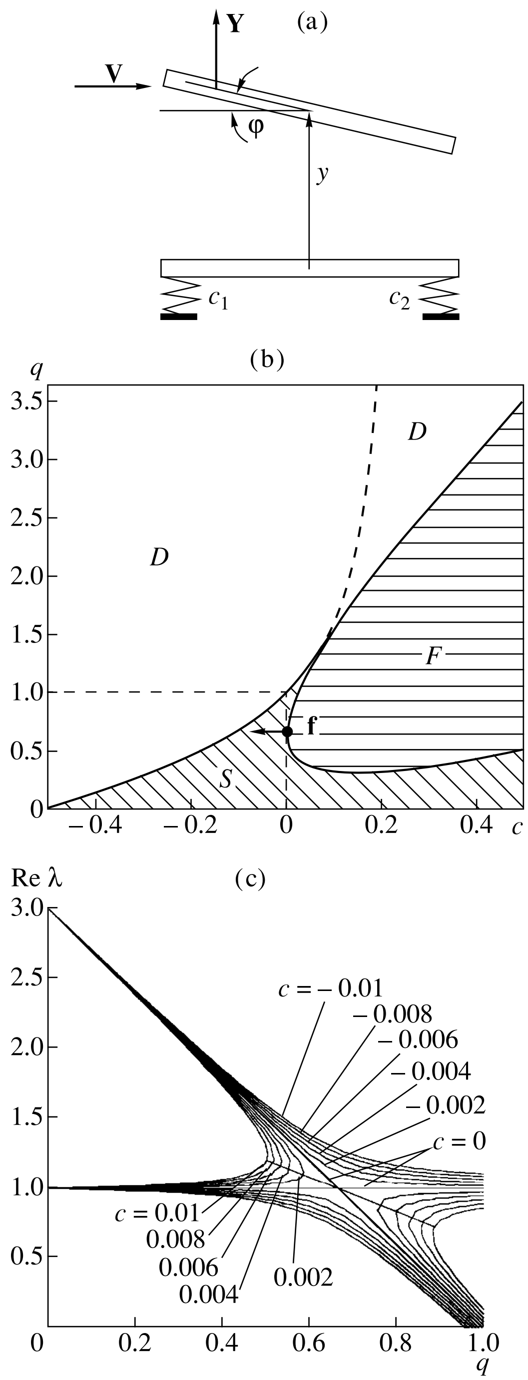

Fig. 2. Stability of vibrations for a plate in a gas flow.

in expansions (8) [10]:

$$\begin{aligned}\lambda &= \lambda_0 + \varepsilon\lambda_2 + \varepsilon^2\lambda_4 + \dots,\\ \mathbf{u} &= \mathbf{u}_0 + \varepsilon\mathbf{w}_2 + \varepsilon^2\mathbf{w}_4 + \dots\end{aligned} \tag{13}$$

The coefficient λ_2 is a solution to the quadratic equation [9]

$$\lambda_2^2 + \lambda_2 a_1 + a_2 = 0 \tag{14}$$

with coefficients

$$\begin{aligned}a_1 &= -(\mathbf{A}_1\mathbf{u}_0, \mathbf{v}_1) - (\mathbf{A}_1\mathbf{u}_1, \mathbf{v}_0),\\ a_2 &= -(\mathbf{A}_2\mathbf{u}_0, \mathbf{v}_0) + (\mathbf{G}_0(\mathbf{A}_1\mathbf{u}_0), \mathbf{A}_1^T\mathbf{v}_0).\end{aligned} \tag{15}$$

The operator $\mathbf{G}_0$, inverse to $\mathbf{A}_0 - \lambda_0\mathbf{I}$, is defined by the expressions

$$(\mathbf{A}_0 - \lambda_0\mathbf{I})\psi = \varphi \quad\text{and}\quad \psi = \mathbf{G}_0\varphi,$$

where φ and $\psi \in C^m$ and the solvability condition $(\varphi, \mathbf{v}_0) = 0$ is assumed to be satisfied. In this case, the solvability condition has the form $(\mathbf{A}_1\mathbf{u}_0, \mathbf{v}_0) = 0$ and is valid for the degenerate directions given by (12) for the vector $\mathbf{e}$. Using expressions (7) for the matrices $\mathbf{A}_1$ and $\mathbf{A}_2$, the coefficients a_1 and a_2 can be written as

$$a_1 = \langle \mathbf{h}, \mathbf{e}_* \rangle, \quad a_2 = \langle \mathbf{H}\mathbf{e}_*, \mathbf{e}_* \rangle - \langle \mathbf{f}, \mathbf{d} \rangle, \tag{16}$$

where the real vector $\mathbf{h}$ and matrix $\mathbf{H}$ are defined by (15).

3. We substitute expressions (16) into Eq. (14) and multiply the result by ε^2. Introducing the notation $\Delta\lambda = \lambda_2\varepsilon$, we have

$$\Delta\lambda^2 + \varepsilon\langle \mathbf{h}, \mathbf{e}_* \rangle\Delta\lambda + \varepsilon^2\langle \mathbf{H}\mathbf{e}_*, \mathbf{e}_* \rangle = \varepsilon^2\langle \mathbf{f}, \mathbf{d} \rangle. \tag{17}$$

It is worth noting that no restrictions were imposed on the vector $\mathbf{d}$. It is convenient to set this vector collinear to the normal $\mathbf{f}$, i.e., to set $\mathbf{d} = \gamma\mathbf{f}$. Denoting $\rho = \gamma|\mathbf{f}|\varepsilon^2$, we see that ε and ρ are coordinates of the vector $\Delta\mathbf{p} = \mathbf{p} - \mathbf{p}_0$ in the orthonormalized basis of the vectors $\mathbf{e}_*$ and $\dfrac{\mathbf{f}}{|\mathbf{f}|}$, which are related by orthogonality condition (12). Indeed,

$$\Delta\mathbf{p} \equiv \varepsilon\mathbf{e}_* + \varepsilon^2\mathbf{d} = \varepsilon\mathbf{e}_* + \rho\frac{\mathbf{f}}{|\mathbf{f}|}.$$

Transforming the right-hand side of Eq. (17), we have

$$\varepsilon^2\langle \mathbf{f}, \mathbf{d} \rangle = \langle \mathbf{f}, \Delta\mathbf{p} \rangle = \rho|\mathbf{f}|. \tag{18}$$

Substituting (18) into (17), we arrive at

$$\Delta\lambda^2 + \varepsilon\langle \mathbf{h}, \mathbf{e}_* \rangle\Delta\lambda + \varepsilon^2\langle \mathbf{H}\mathbf{e}_*, \mathbf{e}_* \rangle = \rho|\mathbf{f}|. \tag{19}$$

This equation describes the splitting of the eigenvalue λ_0, which occurs due to varying the parameters ε and ρ near the stability boundary. In particular, the approximate equation of the stability boundary in the vicinity of the point $\mathbf{p} = \mathbf{p}_0$ follows from (19). Indeed, the boundary between the domains of stability and flutter consists of points that correspond to matrices containing positive double eigenvalues with a Jordan chain of the length of 2. Therefore, the discriminant of quadratic equation (19) has to be zero at the boundary points. This condition leads to a quadratic approximation for the boundary in the local coordinates ε and ρ:

$$\rho = -\varepsilon^2 \frac{D}{4|\mathbf{f}|}, \tag{20}$$

$$D = \langle \mathbf{h}, \mathbf{e}_* \rangle^2 - 4\langle \mathbf{H}\mathbf{e}_*, \mathbf{e}_* \rangle. \tag{21}$$

Parabola (20) is convex either downward (if $D < 0$) or upward (if $D > 0$). It immediately follows from Eq. (19) that the flutter domain is defined by the inequality

$$\rho < -\varepsilon^2 \frac{D}{4|\mathbf{f}|}. \tag{22}$$

If the double eigenvalue λ_0 at the point $\mathbf{p}_0$ is assumed to be negative, then Eq. (20) describes the boundary between the domains of flutter and divergence and inequality (22) defines the flutter domain as before.

For any fixed value of ρ, Eq. (19) describes the behavior of two eigenvalues λ along a line parallel to the tangent to the boundary at the point $\mathbf{p}_0$. Changing the sign of the parameter ρ implies passage through the flutter boundary (see Fig. 1a). It is natural to expect that qualitative changes in the behavior of the frequency curves $\lambda(\varepsilon)$ would take place in this case. Equation (19) allows us to study the rearrangement of the frequency curves both qualitatively and quantitatively, using only information at the point $\mathbf{p} = \mathbf{p}_0$.

We take a perfect square from the left-hand side of (19) and study the cases of positive and negative values of D defined by (21):

$$\left(\Delta\lambda + \frac{1}{2}\varepsilon\langle \mathbf{h}, \mathbf{e}_* \rangle\right)^2 - \left(\frac{\varepsilon\sqrt{D}}{2}\right)^2 = \rho|\mathbf{f}|, \quad D > 0; \tag{23}$$

$$\left(\Delta\lambda + \frac{1}{2}\varepsilon\langle \mathbf{h}, \mathbf{e}_* \rangle\right)^2 + \left(\frac{\varepsilon\sqrt{-D}}{2}\right)^2 = \rho|\mathbf{f}|, \quad D < 0. \tag{24}$$

We then consider Eq. (23) corresponding to the convex flutter domain $D > 0$ (Fig. 1a). If $\rho > 0$, then, for any ε, the eigenvalues λ are real and placed on hyperbola *1* as shown in Fig. 1b. When the parameter ρ tends to zero, the branches of hyperbola *1* approach each other, so that for $\rho = 0$, the eigenvalues are on two real asymptotes:

$$\mathrm{Re}\Delta\lambda(\varepsilon) = -\frac{1}{2}\varepsilon(\langle \mathbf{h}, \mathbf{e}_* \rangle \pm \sqrt{D}).$$

If $\rho < 0$, the set of real solutions to Eq. (23) consists of two branches of adjacent hyperbola *2* in Fig. 1b; however, for ε satisfying the inequality $\varepsilon^2 < -\frac{4\rho|\mathbf{f}|}{D}$, the eigenvalues λ are complex-valued and belong to an ellipse (a so-called instability bubble). The real and imaginary components of the bubble are governed, respectively, by the equations

$$\mathrm{Re}\Delta\lambda = -\frac{\varepsilon}{2}\langle \mathbf{h}, \mathbf{e}_* \rangle, \tag{25}$$

$$(\mathrm{Im}\Delta\lambda)^2 + \left(\frac{\varepsilon\sqrt{D}}{2}\right)^2 = -\rho|\mathbf{f}|. \tag{26}$$

The ellipse and two branches of hyperbola *2* have common points when $\varepsilon_{1,2} = \pm\sqrt{-\frac{4\rho|\mathbf{f}|}{D}}$ (Fig. 1c), at which the matrix $\mathbf{A}$ has the double real eigenvalues

$$\lambda_{1,2} = \lambda_0 \mp \frac{1}{2}\langle \mathbf{h}, \mathbf{e}_* \rangle \sqrt{-\frac{4\rho|\mathbf{f}|}{D}}.$$

Hyperbola (23) and ellipses (25) and (26) lie on the orthogonal planes since the real component of the ellipse is given by a straight line (Fig. 1b). Therefore, when varying the parameter ε, the eigenvalues bifurcate in the vicinity of the points ε_1 and ε_2. Namely, two eigenvalues, moving on a plane, merge and then come out of the plane in the direction orthogonal to it. Such a behavior of the eigenvalues is referred to as a strong interaction and is typical for passage through the flutter boundary [8]. It is worth noting that the inequality determining the instability bubble exactly coincides with the approximation of flutter domain (22). Thereby, the overlapping is closely related to the convexity properties of the flutter domain.

We now consider the case $D < 0$ when the overlapping of the frequency curves is described by the families of ellipses (24). There are no real solutions for $\Delta\lambda(\varepsilon)$ if $\rho < 0$ (the case of flutter). At $\rho = 0$, the set of real solutions consists of an isolated point. If $\rho > 0$, then the set of real solutions $\Delta\lambda(\varepsilon)$ is ellipse (24), whose boundary is defined by the inequality $\varepsilon^2 < \frac{4\rho|\mathbf{f}|}{(-D)}$. It is easy to see that such a behavior corresponds to concave flutter domain (22).

Remark 1. Similarly, using variations along the curves $\mathbf{p}(\varepsilon) = \mathbf{p}_0 + \varepsilon\mathbf{e} + \varepsilon^2\mathbf{d}$, we can analyze the overlapping of the frequency curves near the boundary between the domains of stability and divergence.

Remark 2. The basic equations describing the bifurcations of eigenvalues, (11) and (17), and the overlapping of frequency curves, (23) and (24), are also valid for the linear differential operators $\mathbf{A}$ under homogeneous linear boundary conditions, depending on the parameters. The difference is that the vectors $\mathbf{f}$ and $\mathbf{h}$ and matrix $\mathbf{H}$ are defined through eigenfunctions and associated functions of the operator and through the derivatives of both the differential expression and the boundary forms with respect to parameters.

4. As an example, we consider a simple problem of stability for a rigid plate in an incident gas flow [11]. The plate is fixed by two elastic supports having stiffness coefficients c_1 and c_2 per unit length and has two degrees of freedom: the vertical displacement y and the angle φ of deflection (see Fig. 2a). Small vibrations of the plate are described by the following equations in dimensionless variables [9, 11]:

$$\begin{pmatrix} \ddot{y} \\ \ddot{\varphi} \end{pmatrix} + \mathbf{A}\begin{pmatrix} y \\ \varphi \end{pmatrix} = 0, \quad \mathbf{A} = \begin{bmatrix} 1 & c-q \\ 12c & 3-3q \end{bmatrix}. \quad (27)$$

Here, $q = \frac{1}{2}c_y^a\rho v^2(c_1 + c_2)^{-1}$ is the load parameter proportional to the dynamic pressure of the flow and $c = \frac{1}{2}(c_1 - c_2)(c_1 + c_2)^{-1}$ is the parameter characterizing the relation between the stiffness coefficients. Thus, circulatory system (27) depends on the vector of parameters $\mathbf{p} = (c, q)$. It follows from physical reasons that $q \ge 0$ and $-0.5 \le c \le 0.5$.

Seeking the solution to Eq. (27) in the form $\begin{pmatrix} y \\ \varphi \end{pmatrix} = \mathbf{u}e^{i\omega\tau}$, we arrive at eigenvalue problem (2). With $\lambda = \omega^2$, the corresponding characteristic equation has the form

$$\lambda^2 + (3q-4)\lambda + 12cq - 3q - 12c^2 + 3 = 0. \quad (28)$$

Equations of the curves subdividing the plane of the parameters c and q into the domains of stability, flutter, and divergence follow directly from (28) (see Fig. 2b):

$$q_d(c) = \frac{1-4c^2}{1-4c}, \qquad q_f(c) = \frac{2}{3}(1 + 4c \pm 2\sqrt{c(c+2)}). \quad (29)$$

For the point $\left(c = 0,\ q = \frac{2}{3}\right)$ on the boundary between the domains of stability and flutter, which corresponds to the double eigenvalue $\lambda = 1$ (see Fig. 2b), characteristic equation (28) can be easily transformed into

$$\left(\lambda - 1 + \frac{3}{2}q - 1\right)^2 - \left(-4c + \frac{3}{2}q - 1\right)^2 = -8c - 4c^2. \quad (30)$$

For $c = 0$, Eq. (30) has two solutions:

$$\lambda = 1, \quad \lambda = 3 - 3q. \quad (31)$$

The two lines (31) intersect one another at the point $\left(q = \frac{2}{3}, \lambda = 1\right)$. If $c \ne 0$, then Eq. (30) describes a family of hyperbolas with asymptotes (31). For small $c < 0$ and $0 \le q \le 1$, the solutions $\lambda(q)$ to Eq. (30) belong to the real plane. One of the two eigenvalues remains positive for any q, while another eigenvalue changes its sign at a certain $q_d < 1$ (see Figs. 2b, 2c). Thus, for $c < 0$ and sufficiently large q, system (27) loses its static stability (the case of divergence). Changing the sign of the parameter c results in a transformation of the frequency curves, which is accompanied by the origination of a region of complex-valued eigenvalues. In this case, the system loses its stability at the values of q_f, such that two positive eigenvalues λ merge to form a double eigenvalue with a Jordan chain of the length of 2 (flutter).

We now show that Eq. (30) can be approximated by formula (23), whose coefficients are calculated by using only information on the system at the point $\mathbf{p}_0 = \left(0, \frac{2}{3}\right)$. The eigenvectors and associated vectors of the double eigenvalue $\lambda = 1$ are

$$\mathbf{u}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_0 = \begin{pmatrix} 0 \\ -2/3 \end{pmatrix}, \qquad \mathbf{u}_1 = \begin{pmatrix} 0 \\ -3/2 \end{pmatrix}, \quad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \quad (32)$$

Formulas (10) and (12) determine the vector $\mathbf{f}$ normal to the flutter boundary at the point $\mathbf{p}_0 = \left(0, \frac{2}{3}\right)$ and, therefore, the tangent vector $\mathbf{e}_*$:

$$\mathbf{f} = \begin{pmatrix} -8 \\ 0 \end{pmatrix}, \quad \mathbf{e}_* = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (33)$$

Finding the vector $\mathbf{h}$ and matrix $\mathbf{H}$ by formulas (15) and (16), with regard to expression (21), we obtain that $\langle \mathbf{h}, \mathbf{e}_* \rangle = 3$, $D = 9$, and the variation vector is $\Delta\mathbf{p} = \left(c, q - \frac{2}{3}\right)$. Thus, the parameter ρ in Eqs. (23) and (24) is $\frac{\langle \mathbf{f}, \Delta\mathbf{p} \rangle}{|\mathbf{f}|} = -c$. As $D > 0$, the transformation of the frequency curves is described by Eq. (23), which in this case takes the form

$$\left(\lambda - 1 + \frac{3}{2}q - 1\right)^2 - \left(\frac{3}{2}q - 1\right)^2 = -8c. \quad (34)$$

Comparing exact equation (30) with its approximation (34), we see that the asymptotes $\lambda = 1$ and $\lambda = 3 - 3q$ coincide completely in both equations, with the approximations of the frequency curves being rather good for small values of parameter c. The quadratic approximation of the flutter domain in the vicinity of the point $\mathbf{p}_0 = \left(0, \frac{2}{3}\right)$ is given by formula (22):

$$c > \frac{9}{32}\left(q - \frac{2}{3}\right)^2. \quad (35)$$

Approximation (35) implies that the flutter domain is convex (see Fig. 2b), and it is in good agreement with the exact expression $q_f(c)$ for flutter boundary (29).

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