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Solution to the Herrmann–Smith Problem[1]

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In 1958, a problem was posed by Dzhanelidze [1] on the stability of a cantilever column compressed at the free end by a force inclined at a certain angle to the tangent of the elastic line (partially following force). It was shown that, depending on the values of problem parameters, this nonconservative system is subjected to both static (divergence) and dynamic (flutter) instability. Kordas and Życzkowski [2] have thoroughly analyzed the boundaries of stability and instability domains in this problem. Furthermore, Smith and Herrmann [3] have generalized the problem by considering a column attached to the Winkler elastic foundation. They found that the critical load causing the flutter does not depend on the modulus of the uniform elastic foundation. This effect, known as the Herrmann–Smith paradox, has stimulated considerable interest and many publications throughout the world [4–9].

In the present paper, the complete solution to the Herrmann–Smith problem is given. The effect of the elastic-base modulus κ on the stability, flutter, and divergence domains in the plane of parameters (follower force p–deviation angle η) is analyzed. It turned out that the increase in the elastic-foundation modulus shifts the static-instability domain away expanding the stability domain. Since the flutter boundary of the circulatory system is determined by the multiple eigenvalues with the Keldysh chain, the derivative of the critical flutter load with respect to κ at the flutter boundary is zero, whence follows the invariance of the flutter boundary with respect to the variation of the parameter κ. The explicit expression for the gradient of the critical flutter load with respect to the distribution of the elastic-foundation modulus $\kappa(x)$ along the column length is obtained. It is shown that the gradient function has alternating sign. Therefore, the rearrangement of the elastic-foundation modulus $\kappa(x)$ along the column may lead to both an increase and a decrease in the critical flutter load. It turned out that within the range $0.6 \lesssim \eta \leq 1$, the free end of the column is the most sensitive to the change of the uniformity of $\kappa(x)$ with respect to the critical load.

[1] The article was submitted by the authors in English.

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1. STABILITY DIAGRAM FOR A COLUMN ATTACHED TO AN ELASTIC FOUNDATION

We consider a uniform elastic cantilever column of length L attached to an elastic Winkler base having the constant modulus of rigidity χ (Fig. 1). It is assumed that the free end of the column is loaded by a non-conservative force P whose direction is determined by the parameter η. The case $\eta = 1$ implies that the column is loaded by a purely tangential follower force (Beck's problem). If $\eta = 0$, then the force P is potential (conservative).

We consider plane transverse vibrations of the column introducing dimensionless variables

$$x = \frac{X}{L}, \quad y = \frac{Y}{L}, \quad \tau = t\sqrt{\frac{EI}{\rho A L^4}},$$

$$q = \frac{PL^2}{EI}, \quad \kappa = \frac{\chi L^4}{EI},$$

where t is time, ρ is the column material density, A is the cross-section area, E is the Young modulus, and I is the

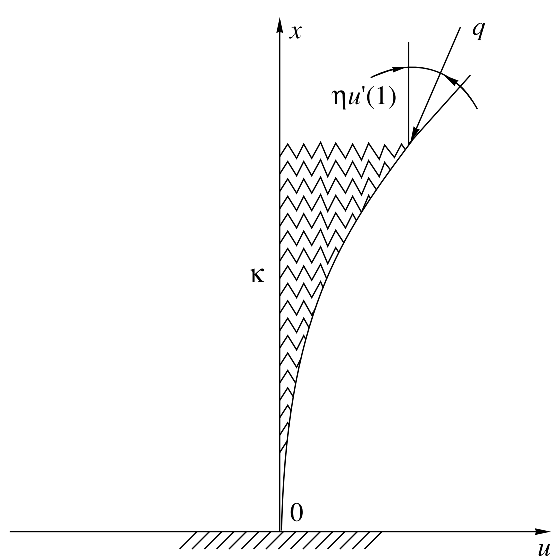

Fig. 1. Column installed on an elastic foundation and loaded by the tangential follower force.

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moment of inertia of the column cross section. The differential equation describing small column vibrations and the appropriate boundary conditions have the form

$$\begin{gathered} y''''(x,\tau) + qy''(x,\tau) + \kappa y(x,\tau) + \ddot{y}(x,\tau) = 0, \\ y(0,\tau) = y'(0,\tau) = y''(1,\tau) \\ = y'''(1,\tau) + (1-\eta)qy'(1,\tau) = 0. \end{gathered} \tag{1}$$

Here, dots and primes imply differentiation with respect to time τ and coordinate x, respectively. Isolating time in the relationship $y(x, \tau) = u(x)\exp(i\sqrt{\lambda}\tau)$, we arrive at the eigenvalue problem [3]

$$\begin{gathered} u'''' + qu'' + \kappa u = \lambda u, \quad u(0) = u''(1) = u'(0) \\ = u'''(1) + (1-\eta)qu'(1) = 0. \end{gathered} \tag{2}$$

The adjoint problem has the form

$$\begin{gathered} v'''' + qv'' + \kappa v = \lambda v, \quad v(0) = v''(1) + \eta q v(1) \\ = v'(0) = v'''(1) + qv'(1) = 0. \end{gathered} \tag{3}$$

Eigenfunctions for adjoint eigenvalue problems (2) and (3) are

$$\begin{gathered} u(x) = \cosh(ax) - \cos(bx) + \frac{a^2\cosh a + b^2\cos b}{ab(a\sinh a + b\sin b)} \\ \times(a\sin(bx) - b\sinh(ax)), \end{gathered} \tag{4}$$

$$\begin{gathered} v(x) = \cosh(ax) - \cos(bx) \\ + \frac{(a^2+\eta q)\cosh a + (b^2 - \eta q)\cos b}{b(a^2+\eta q)\sinh a + a(b^2-\eta q)\sin b} \\ \times(a\sin(bx) - b\sinh(ax)), \end{gathered} \tag{5}$$

$$\begin{gathered} a = \sqrt{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \lambda - \kappa}}, \quad b = \sqrt{\frac{q}{2} + \sqrt{\frac{q^2}{4} + \lambda - \kappa}}, \\ \kappa \neq \frac{q^2}{4} + \lambda. \end{gathered} \tag{6}$$

Eigenvalues λ are solutions to the characteristic equation

$$\begin{gathered} \Delta(\eta, q, \lambda - \kappa) \equiv (2(\lambda - \kappa) + (1-\eta)q^2) \\ \times(1 + \cosh a\cos b) \\ + q(2\eta - 1)(q + ab\sinh a\sin b) = 0. \end{gathered} \tag{7}$$

Solution of the boundary value problem (1) is stable if and only if all the eigenvalues λ of eigenvalue problem (2) are positive and semisimple. If for all $\lambda \in \mathbb{R}$, some of them are negative, then the system is statically unstable (the case of divergence). The existence of at least one $\lambda \in \mathbb{C}$ implies the dynamic instability (flutter).

For given values of the parameters $\eta = \eta_0$, $q = q_0$ and real-valued λ, zeros of the function $\Delta(\lambda - \kappa)$ differ from zeros of $\Delta(\lambda)$ by the quantity κ, while their multiplicities remain the same. Since the flutter boundary is defined by the multiple real-valued eigenvalues, the point (κ, η_0, q_0) belongs to the flutter boundary for arbitrary κ if the point $(0, \eta_0, q_0)$ also belongs to this boundary. The boundary between the stability and divergence domains determined by zero eigenvalues has to change while varying the modulus κ of the elastic foundation, since the eigenvalue $\lambda_0 = 0$ becomes nonzero after the shift $\lambda = \kappa$.

We now find the divergence domain in the Herrmann–Smith problem. Substituting $\lambda = 0$ into both characteristic equation (7) and expressions (6), we obtain

$$\begin{gathered} \eta(q,\kappa) = \frac{r_2 q - r_1(q^2 - 2\kappa)}{2r_2 q - r_1 q^2}, \quad r_1 = 1 + \cosh a\cos b, \\ r_2 = q + ab\sinh a\sin b. \end{gathered} \tag{8}$$

The function $\eta(q, 0)$ attains its maximum $\eta_* = \frac{1}{2}$ at $q_* = (2j+1)^2\pi^2$, $j = 0, 1, \ldots$, whereas $\eta(q, 0) \longrightarrow -\infty$ at the points $q_{**} = (2j)^2\pi^2$ [2]. For arbitrary $\kappa > 0$, the function $\eta(q)$ $(q > 0)$ defined in (8) represents a smooth curve of zero eigenvalues on the plane of the parameters η and q. In this case, the maximum value of η_* and the corresponding value of the parameter q_* change when varying the parameter κ. The numerical solution to the equation $\frac{d\eta}{dq} = 0$ at different values of κ yields the function $q_*(\kappa)$ and [with the use of (8)] $\eta_*(\kappa)$. Knowledge of these functions allows us to find the trajectory in the plane of the parameters η and q of the point $\mathbf{p}_* = (\eta_*, q_*)$ that corresponds, e.g., to the first maximum associated with a change in the parameter κ. The point of the first maximum of the function $\eta(q)$ with an increase in κ moves along the q axis oscillating and tending to the line $\eta = \frac{1}{2}$ as $\kappa \longrightarrow \infty$. It can be shown that all other maxima of the function $\eta(q)$ manifest the same behavior. Therefore, for $\eta > \frac{1}{2}$, the divergence is absent for any $\kappa \geq 0$.

Figure 2 demonstrates stability diagrams for $\eta \in [0, 1]$ and $q \in [0, 150]$. As is seen, within this range of parameters, there exist two curves for double real-valued eigenvalues invariable with respect to the change of the parameter κ. Certain parts of these curves form the flutter boundary (F). The lower curve corresponds to the confluence of the first two eigenvalues, while the

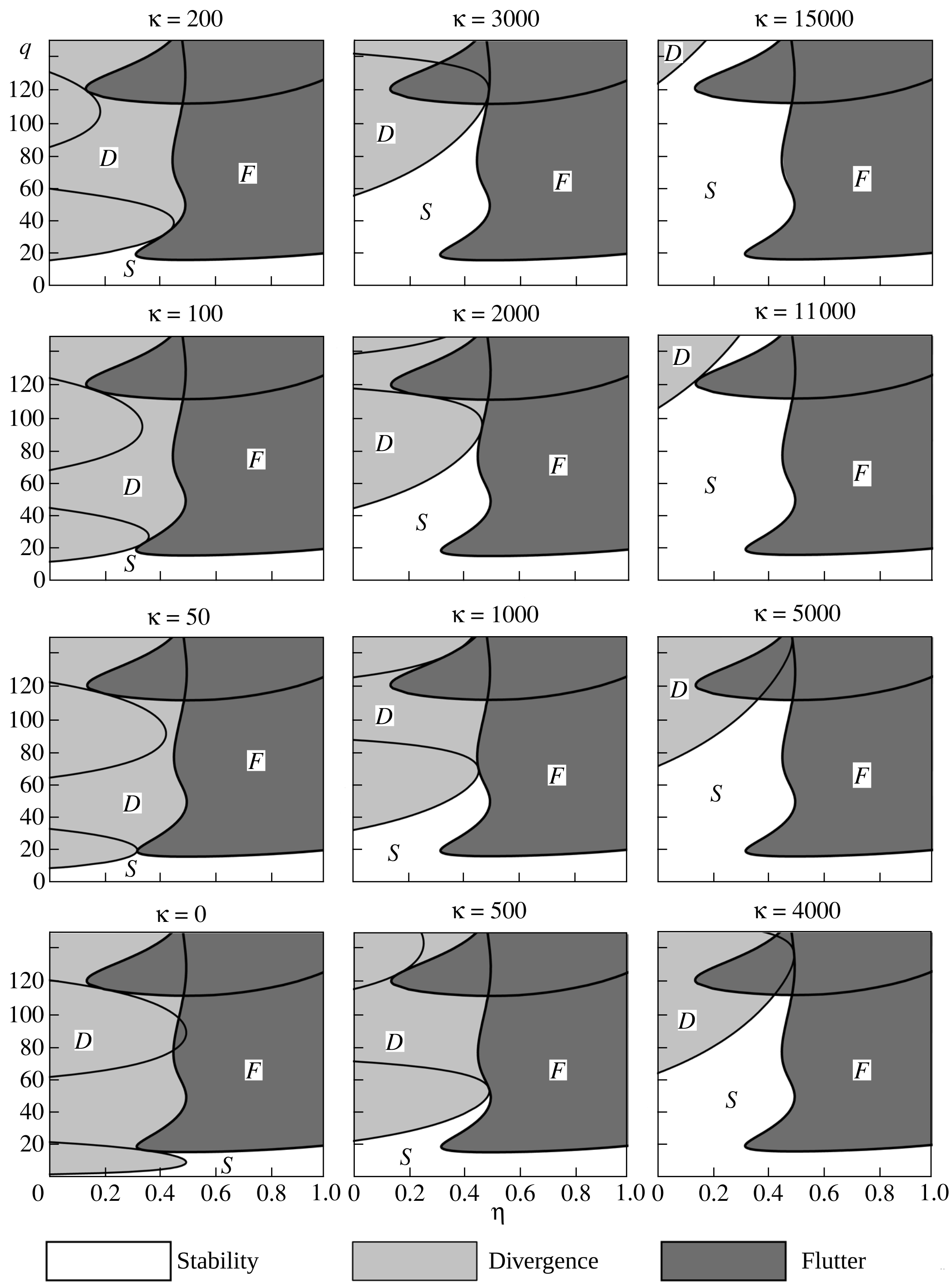

Fig. 2. Shift of the divergence domain due to an increase in the base modulus of rigidity κ.

upper curve corresponds to the coincidence of the third and fourth eigenvalues. In the region of Fig. 2 corresponding to $\kappa < 3000$, there also exist two curves of zero eigenvalues. At the points of the upper curve, the third and the fourth eigenvalues change their sign. The second eigenvalue passes through the zero at the points of the upper part of the lower curve. The part of the lower curve in which the first eigenvalue changes its sign forms the boundary between the stability and divergence domains.

In [2, 10] it was shown that for $\kappa = 0$ at the flutter boundary, there exists a point $\mathbf{p}_0 = (0.35431330, 17.0695748)$ corresponding to the double zero eigenvalue (0^2) at which the curve of the simple zero eigenvalues is tangent to the flutter boundary, Fig. 2. With the increase of the modulus of the elastic foundation, the flutter boundary preserves its form while the double real-valued eigenvalues linearly change with the variation of κ. As a consequence, the double zero eigenvalue moves along the flutter boundary. Since the curve of the zero eigenvalues and the flutter boundary have the same tangent at the point 0^2, the curves of the zero eigenvalues and the divergence domain travel together with them in the plane of the parameters η and q with the increase in κ clearing the place for the stability domain. This process is reproduced in Fig. 2 for $\kappa \in [0, 15000]$. In the case of $\kappa > 3000$, only the lowest curve of the zero eigenvalues remains within the range $\eta \in [0, 1]$, $q \in [0, 150]$. At $\kappa = 15000$, the divergence domain almost completely goes out of this range, and we have in this region a vast stability domain, as shown in Fig. 2.

Of all publications devoted to the Herrmann–Smith problem, probably only in [8] was the question of the change of the stability diagram in the plane of the parameters η, q while varying the modulus κ of the elastic foundation touched upon. However, the authors of [8] did not find the necessary curves in the parameter plane and omitted the phenomenon of the divergence domain traveling along the flutter boundary. In [2], the stability diagram for $\kappa = 0$ was found without the fragment of the flutter boundary, which corresponds to the negative double eigenvalues since it does not correspond to any physical interpretation [2]. As is seen from Fig. 2, introducing the parameter κ considerably changes the problem, transforming this curve segment into the boundary between the flutter domain and stability domain.

2. SENSITIVITY OF THE CRITICAL LOAD TO THE NONHOMOGENEITY OF AN ELASTIC FOUNDATION

We consider the point $\mathbf{p}_0 = (\kappa_0, \eta_0, q_0)$ on the flutter boundary, which corresponds to the double real eigenvalue λ_0 with the Keldysh chain of length 2. The chain consists of the eigenfunction u_0 and the associated function u_1. The adjoint eigenfunction and associated function of the eigenvalue λ_0 are denoted by v_0 and v_1, respectively. The functions u_0, u_1, v_0, and v_1 can be chosen real-valued. Since the eigenvalue λ_0 has the Keldysh chain, the eigenfunctions u_0 and v_0 are orthogonal: $\int_0^1 u_0 v_0 dx = 0$, [10].

In [3], it was shown that the critical flutter load q_0 in the Herrmann–Smith problem does not depend on the modulus of the uniform elastic foundation κ. We now study the sensitivity of this load with respect to the small inhomogeneity of the base. In doing this, we consider the variation of the rigidity of the elastic foundation $\kappa(x) = \kappa_0 + \delta\kappa(x)$, where $\delta\kappa(x) = \epsilon e_\kappa(x)$ and $\epsilon \geq 0$ is a small parameter. In that case, the parameters η and q take the increments

$$\eta = \eta_0 + \epsilon e_\eta + o(\epsilon), \quad q = q_0 + \epsilon e_q + o(\epsilon).$$

Substituting these variations into the eigenvalue problem (2) and taking into account the expansions

$$\lambda = \lambda_0 + \epsilon^{1/2}\lambda_1 + \epsilon\lambda_2 + \epsilon^{3/2}\lambda_3 + \ldots,$$
$$u = u_0 + \epsilon^{1/2}w_1 + \epsilon w_2 + \epsilon^{3/2}w_3 + \ldots,$$

valid for the double eigenvalue, we arrive at the boundary value problems determining the first coefficients to these expansions:

$$\begin{gathered} w_1'''' + q_0 w_1'' + \kappa_0 w_1 = \lambda_0 w_1 + \lambda_1 u_0, \\ w_1(0) = w_1'(0) = w_1''(1) \\ = w_1'''(1) + (1-\eta_0) q_0 w_1'''(1) = 0; \end{gathered} \tag{9}$$

$$\begin{gathered} w_2'''' + q_0 w_2'' + \kappa_0 w_2 \\ = \lambda_0 w_2 + \lambda_1 w_1 + \lambda_2 u_0 - e_q u_0'' - e_\kappa(x) u_0, \\ w_2(0) = w_2'(0) = w_2''(1) \\ = w_2'''(1) + (1-\eta_0) q_0 w_2'''(1) \\ + [e_q(1-\eta_0) - e_\eta q_0] u_0'(1) = 0. \end{gathered} \tag{10}$$

After taking the scalar product of Eq. (10) and the eigenfunction v_0 and substituting the solution w_1 of problem (9) into the resulting product, we found the approximate formula describing the splitting of the double real-valued eigenvalue λ_0

$$\lambda = \lambda_0 \pm \sqrt{\frac{(\delta\kappa(x)u_0, v_0)}{(u_0, v_1)} + \frac{q_0 u_0'(1) v_0(1)}{(u_0, v_1)}\Delta\eta + \frac{(u_0'', v_0) - (1-\eta_0)u_0'(1)v_0(1)}{(u_0, v_1)}\Delta q}. \tag{11}$$

Here, $\Delta\eta = \eta - \eta_0$, $\Delta q = q - q_0$ and $(\phi, \psi) = \int_0^1 \phi\bar{\psi} dx$ denotes the scalar product. In the first approximation, the double eigenvalue λ_0 splits due to the variation of the parameters into two simple eigenvalues (stability or divergence depending on the sign of λ_0) if the radicand in formula (11) is positive. If the radicand is negative, then the double eigenvalue λ_0 splits into the complexadjoint pair (flutter). Zero radicand in (11) implies the absence of splitting of λ_0. The corresponding condition can be written in the form

$$\Delta q = -\frac{(\delta\kappa(x)u_0, v_0)}{(u_0'', v_0) - (1-\eta_0)u_0'(1)v_0(1)} - \frac{q_0 u_0'(1)v_0(1)}{(u_0'', v_0) - (1-\eta_0)u_0'(1)v_0(1)}\Delta\eta. \quad (12)$$

Formula (12) represents the linear part of the increment of the critical flutter load q_0 while changing the parameter η and the rearrangement of the rigidity of the elastic foundation κ along the column. If the base modulus increases uniformly so that $\delta\kappa$ = const, then

$$\int_0^1 \delta\kappa u_0 v_0 dx = \delta\kappa\int_0^1 u_0 v_0 dx = 0$$

and the flutter load does not depend on the rigidity κ of the uniform elastic foundation.

Introducing the function of the gradient of the critical flutter load with respect to the distribution of the elastic-base modulus $\kappa(x)$

$$g(x) = \frac{-u_0(x)v_0(x)}{\int_0^1 u_0'' v_0 dx - (1-\eta_0)u_0'(1)v_0(1)}, \quad (13)$$

we rewrite formula (12) in the form

$$\Delta q = \int_0^1 g(x)\delta\kappa(x)dx + \frac{\partial q}{\partial\eta}\Delta\eta,$$

$$\frac{\partial q}{\partial\eta} = \frac{-q_0 u_0'(1)v_0(1)}{\int_0^1 u_0'' v_0 dx - (1-\eta_0)u_0'(1)v_0(1)}. \quad (14)$$

We consider, e.g., the point (κ_0 = const, $\eta_0 = 1$, $q_0 = 20.0509536$) on the boundary between the flutter domain and stability domain corresponding to the purely tangential follower force. Substituting into (13) the eigenfunctions $u_0(x)$ and $v_0(x)$ evaluated at this point with the use of (4)–(6), we find the gradient function $g(x)$. From Fig. 3, it is seen that the gradient is an oscillating function. Hence, for $\eta_0 = 1$, the column's free end is the most sensitive to a variation of the modulus κ.

We now vary the parameter κ in the gradient direction: $\delta\kappa(x) = \gamma g(x)$. Assuming the parameter η_0 to be fixed and substituting $\delta\kappa(x)$ into (14), we find the approximate expression for the critical flutter load for $\eta_0 = 1$ and $q_0 = 20.0509536$:

$$q = q_0 + \gamma\int_0^1 g^2(x)dx = q_0 + \gamma \cdot 0.00708579.$$

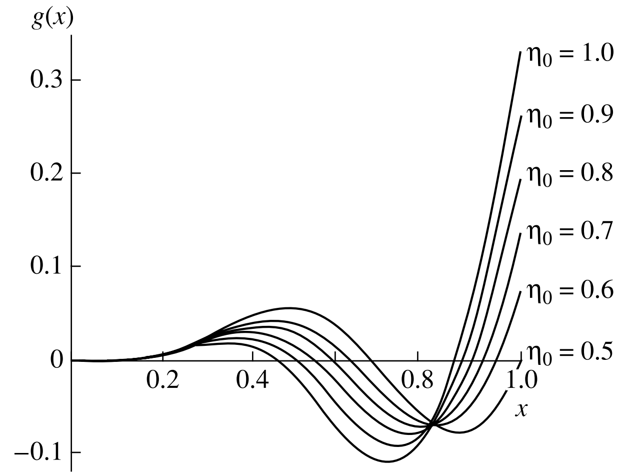

Fig. 3. Gradient function $g(x)$ of the critical flutter load with respect to $\kappa(x)$.

The formula obtained shows that the violation of the uniformity of the elastic foundation can both increase ($\gamma > 0$) and decrease ($\gamma < 0$) the critical flutter load.

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