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MATHEMATICAL PHYSICS

Berry Phase around Degeneracies

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When a quantum system adiabatically traverses a closed cycle in the parameter space, the wave function of the system acquires a complex factor depending only on the cycle. This phenomenon is known as the Berry or geometric phase [1]. The Berry phase has been addressed in hundreds of studies, whose results were summarized in [2–5]. A similar effect occurs in optics, wave dynamics, and mechanics [4, 6, 7]. The Berry phase is closely related to energy level degeneracies: in the neighborhood of degeneracies, the phase can be finite for an arbitrarily small cycle in the parameter space.

In this paper, general asymptotic expressions are found for the Berry phase in the neighborhood of double degeneracies. We consider both the classical case of stationary degeneracies and the case of quasi-stationary (metastable) degeneracies. In conclusion, we discuss applications to optical and mechanical systems. The results presented are based on the bifurcations theory of multiple eigenvalues [8, 9] and on the theory of versal deformations [10] and correspond to V.I. Arnold's suggestion that a theory of residues be developed for calculating the Berry phase [11].

1. Let $H(X)$ be the Hamiltonian of a quantum system, which depends on the vector $X = (X_1, X_2, \ldots, X_m)$ of m real parameters. This is a linear Hermitian operator on the space of states (wave functions); i.e., $H^\dagger = H$, where $H^\dagger$ is an adjoint operator. Stationary bounded states are determined by the eigenvectors $|\psi_n(X)\rangle$ of the discrete spectrum of the Hamiltonian, and the (real) eigenvalues $E_n(X)$ correspond to the energy levels, where n is the level index (here and below, we use the Dirac notation). It is assumed that the eigenvectors corresponding to nondegenerate energy levels are chosen uniquely and depend smoothly on the parameters. Consider the evolution of the system as the parameter vector $X(t)$ undergoes an adiabatic cyclic variation; i.e., $X(0) = X(T)$. Denote by $C = \{X(t): 0 \le t \le T\}$ the parameter variation trajectory, which is an oriented cycle. At the initial time $t = 0$, let the system be at the nondegenerate discrete level $E_n = E_n(X(0))$; i.e., the wave function $|\Psi(t)\rangle$ at $t = 0$ be $|\Psi(0)\rangle = |\psi_n(X(0))\rangle$. As a result of the cyclic variation in the parameters, the system returns to the initial position (in the adiabatic approximation), acquiring a phase factor consisting of the dynamic phase $\delta_n = -\frac{1}{\hbar}\int_0^T E_n(t)dt$ and the geometric (Berry) phase γ_n: $|\Psi(T)\rangle = e^{i(\delta_n + \gamma_n)}|\Psi(0)\rangle$. The Berry phase is determined by the following integral along C in the parameter space [1]:

$$\gamma_n = i\oint_C \frac{\langle \psi_n(X)|d\psi_n(X)\rangle}{\langle \psi_n(X)|\psi_n(X)\rangle}. \quad (1)$$

Formula (1) simplifies after applying the standard normalization $\langle\psi_n(X)|\psi_n(X)\rangle = 1$. Note that γ_n is a real number for Hermitian Hamiltonians.

Double degeneracies arise when neighboring energy levels coincide: $E_n = E_{n+1}$. It is well known that such degeneracies in the parameter space have codimension 2 and 3 for real and complex Hamiltonians, respectively (which describe reversible and irreversible systems). Let X_{DP} be the point in the parameter space that corresponds to the double degeneracy $E_{\mathrm{DP}} = E_n(X_{\mathrm{DP}}) = E_{n+1}(X_{\mathrm{DP}})$. This point is called a diabolic point (DP) [12]. The eigenvectors $|\psi_n^{\mathrm{DP}}\rangle = |\psi_n(X_{\mathrm{DP}})\rangle$ and $|\psi_{n+1}^{\mathrm{DP}}\rangle = |\psi_{n+1}(X_{\mathrm{DP}})\rangle$ at the diabolic point are chosen so that they satisfy the orthogonality condition $\langle\psi_n^{\mathrm{DP}}|\psi_{n+1}^{\mathrm{DP}}\rangle = 0$ and the normalization conditions $\langle\psi_n^{\mathrm{DP}}|\psi_n^{\mathrm{DP}}\rangle = \langle\psi_{n+1}^{\mathrm{DP}}|\psi_{n+1}^{\mathrm{DP}}\rangle = 1$.

Consider small cycles of the form $C = \{X(t) = X_{\mathrm{DP}} + \varepsilon\hat{X}(t): 0 \le t \le T\}$ around X_{DP}. Here, $\hat{X}(T) = \hat{X}(0)$ and

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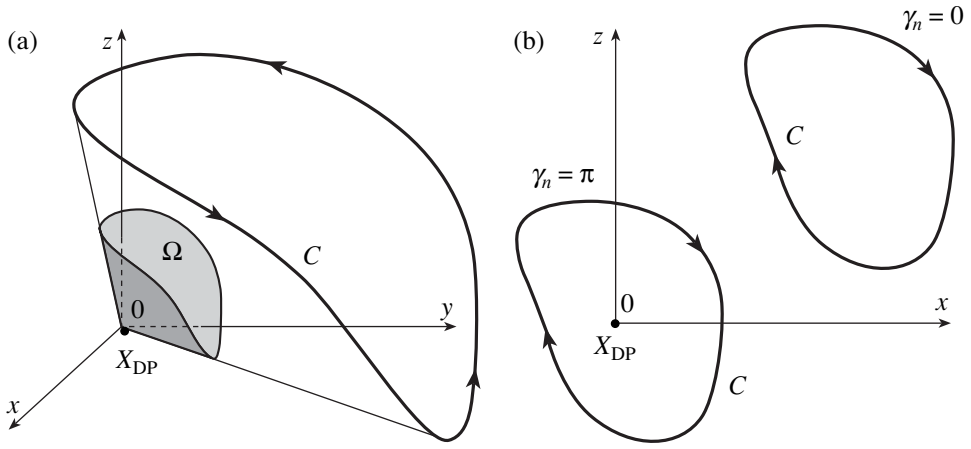

Fig. 1. Berry phase for a cycle around a diabolic point for (a) complex and (b) real Hamiltonians.

$\varepsilon > 0$ is a small parameter. We use the following asymptotic formulas for the eigenvalues and eigenvectors [8, 9]:

$$\begin{aligned} E_{n,n+1}(X(t)) &= E_{DP} + \varepsilon\mu_{n,n+1} + o(\varepsilon), \\ |\psi_{n,n+1}(X(t))\rangle &= \alpha_{n,n+1}|\psi_n^{DP}\rangle + \beta_{n,n+1}|\psi_{n+1}^{DP}\rangle + o(1), \end{aligned} \tag{2}$$

where $\mu_{n,n+1}$, $\alpha_{n,n+1}$, and $\beta_{n,n+1}$ are scalars determined by the eigenvalue equation for the Hermitian matrix

$$\begin{pmatrix} \langle\psi_n^{DP}|H_1|\psi_n^{DP}\rangle - \mu_{n,n+1} & \langle\psi_n^{DP}|H_1|\psi_{n+1}^{DP}\rangle \\ \langle\psi_{n+1}^{DP}|H_1|\psi_n^{DP}\rangle & \langle\psi_{n+1}^{DP}|H_1|\psi_{n+1}^{DP}\rangle - \mu_{n,n+1} \end{pmatrix} \times \begin{pmatrix} \alpha_{n,n+1} \\ \beta_{n,n+1} \end{pmatrix} = 0. \tag{3}$$

Here, εH_1 is the linear part of the parameter dependence of the Hamiltonian at X_{DP}:

$$H_1(X(t)) = \sum_{k=1}^{m} \frac{\partial H}{\partial X_k}\bigg|_{X = X_{DP}} \hat{X}_k(t). \tag{4}$$

We introduce real variables x, y, and z related to the parameter vector $X = X_{DP} + \hat{X}$ by the linear relations

$$\begin{aligned} x + iy &= \langle\psi_{n+1}^{DP}|H_1|\psi_n^{DP}\rangle, \\ z &= \frac{\langle\psi_n^{DP}|H_1|\psi_n^{DP}\rangle - \langle\psi_{n+1}^{DP}|H_1|\psi_{n+1}^{DP}\rangle}{2}. \end{aligned} \tag{5}$$

Then the energy levels around the degeneracy are determined by (2) and (3) in the form

$$E_{n,n+1} = E_{DP} + \varepsilon\left(\frac{\langle\psi_n^{DP}|H_1|\psi_n^{DP}\rangle}{2} + \frac{\langle\psi_{n+1}^{DP}|H_1|\psi_{n+1}^{DP}\rangle}{2} \pm \sqrt{x^2 + y^2 + z^2}\right) + o(\varepsilon), \tag{6}$$

where, if $E_n \le E_{n+1}$, the minus and plus signs correspond to the indices n and $n + 1$, respectively. It can be seen from (6) that x, y, and z are effective parameters responsible for the unfolding of the degeneracy.

By using (2)–(5) in (1) and switching to the spherical coordinates $x = r\sin\theta\cos\varphi$, $y = r\sin\theta\sin\varphi$, and $z = r\cos\theta$, we find an asymptotic expression for the Berry phase of the nth and $(n + 1)$st states:

$$\begin{aligned} \gamma_{n,n+1} &= \frac{1}{2}\oint_C (1 \pm \cos\theta)d\varphi + o(1) \\ &= \mp\frac{\Omega}{2} + o(1) (\mathrm{mod}\, 2\pi), \end{aligned} \tag{7}$$

where the upper and lower signs correspond to γ_{n+1} and γ_n, respectively. Here, the integral is evaluated along the cycle in the space of x, y, and z that is the image of C under linear transformation (5) and Ω is the solid angle subtended by C on the unit sphere in the space of x, y, and z, (see Fig. 1a, which shows a positively oriented cycle and a diabolic point at the origin). In the zeroth approximation with respect to ε, the phases γ_n and γ_{n+1} have opposite signs. The general expression (7) for the Berry phase agrees with the results obtained for particular physical systems and for Hamiltonians approximable by 2×2 matrices [3].

For reversible systems, the Hamiltonian $H(X)$ is a real operator and the codimension of a double degeneracy is two. In this case, the eigenvectors $|\psi_n^{DP}\rangle$ and $|\psi_{n+1}^{DP}\rangle$ can be chosen to be real. Therefore, $y \equiv 0$ and the image of C in the space of x, y, and z lies in the plane (x, z). As a result, we obtain the well-known expressions for the phase: $\gamma_{n,n+1} = \pi$ if C makes a single turn around the diabolic point ($\Omega = 2\pi$) and $\gamma_{n,n+1} = 0$ if the diabolic point lies outside the cycle ($\Omega = 0$) (Fig. 1b). This result does not contain the small component $o(1)$ since the Berry phase for reversible systems depends only on the topology of the cycle with respect to the degeneracies [3].

Indeed, in the case of a real Hamiltonian, integral (1) transforms into $\gamma_n = i\oint_C \frac{d\ln\langle\overline{\psi_n(X)}|\psi_n(X)\rangle}{2}$; i.e., it depends only on the topology of C with respect to the degeneracy.

Note that the real eigenvector $|\psi_n(X)\rangle$ reverses its sign while going around a degeneracy [11]. Thus, for the eigenvector to be uniquely determined on the cycle, a complex factor has to be introduced that varies from 1 to –1 over a single turn around X_{DP}.

2. Quasi-stationary (unstable) states of a quantum system arise as the eigenvectors $|\psi_n(X)\rangle$ of a non-Hermitian Hamiltonian ($H^\dagger \neq H$) that correspond to complex eigenvalues $E_n(X)$ [13]. The real part of an eigenvalue determines the mean energy of a state, and the imaginary part determines the characteristic life time of the state. Generally, the Hamiltonian becomes non-Hermitian due to the boundary condition on the wave function, which requires that the wave be divergent at infinity (instead of vanishing as in the classical condition). Unstable states are frequently called Gamov states. Non-Hermitian terms also arise in effective Hamiltonians in scattering theory [13].

The degenerate states $E_n = E_{n+1}$ of non-Hermitian Hamiltonians have a remarkable property: in contrast to stationary states, their eigenvectors can coincide: $|\psi_n\rangle = |\psi_{n+1}\rangle$. As a result, a double eigenvalue forms a Jordan block, a phenomenon possible only in non-Hermitian systems. The codimension of this phenomenon in the parameter space is two. Note that non-Hermitian systems can involve degeneracies without forming a Jordan block, but their codimension is much higher [9], and they are not considered in this paper.

Let X_{EP} be the point in the parameter space that corresponds to the double degeneracy $E_{\mathrm{EP}} = E_n(X_{\mathrm{EP}}) = E_{n+1}(X_{\mathrm{EP}})$. This point is called an exceptional point (EP) [14]. The eigenvalue E_{EP} corresponds to a single eigenvector $|\chi_0^{\mathrm{EP}}\rangle = |\psi_n(X_{\mathrm{EP}})\rangle = |\psi_{n+1}(X_{\mathrm{EP}})\rangle$ and an associated vector $|\chi_1^{\mathrm{EP}}\rangle$ that satisfy the equations

$$\begin{aligned} H(X_{\mathrm{EP}})|\chi_0^{\mathrm{EP}}\rangle &= E_{\mathrm{EP}}|\chi_0^{\mathrm{EP}}\rangle, \\ H(X_{\mathrm{EP}})|\chi_1^{\mathrm{EP}}\rangle &= E_{\mathrm{EP}}|\chi_1^{\mathrm{EP}}\rangle + |\chi_0^{\mathrm{EP}}\rangle. \end{aligned} \tag{8}$$

The left eigen- and associated vectors $\langle\tilde{\chi}_0^{\mathrm{EP}}|$ and $\langle\tilde{\chi}_1^{\mathrm{EP}}|$ calculated for the adjoint operator $H^\dagger(X_{\mathrm{EP}})$ are defined in a similar fashion. The eigenvectors possess the orthogonality property $\langle\tilde{\chi}_0^{\mathrm{EP}}|\chi_0^{\mathrm{EP}}\rangle = 0$. The normalization conditions are $\langle\tilde{\chi}_0^{\mathrm{EP}}|\chi_1^{\mathrm{EP}}\rangle = \langle\tilde{\chi}_1^{\mathrm{EP}}|\chi_0^{\mathrm{EP}}\rangle = 1$ and $\langle\tilde{\chi}_1^{\mathrm{EP}}|\chi_1^{\mathrm{EP}}\rangle = 0$.

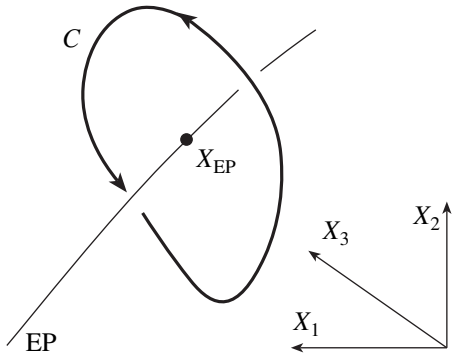

Fig. 2. Cycle around an exceptional point in the parameter space.

Consider the evolution of the system as its parameters adiabatically traverse a small single cycle $C = \{X(t) = X_{\mathrm{EP}} + \varepsilon\hat{X}(t)\colon 0 \le t \le T\}$ around the set of exceptional points (Fig. 2). Let the system be initially at a quasi-stationary state $|\psi_n(X(0))\rangle$. (Here, the adiabatic condition is supplemented with the requirement that the cycle period T be small compared with the life time of the unstable state.) While traversing the cycle in the parameter space, the system arrives at the second interacting state $|\psi_{n+1}(X(0))\rangle$ (up to a complex factor) and returns to the initial state only after making the second cycle [14]. After making two cycles, the wave function acquires a dynamic phase and a geometric (Berry) phase (both of which are generally complex). For the double cycle (denoted as $2C$), the Berry phase is defined as [15]

$$\gamma_n = \gamma_{n+1} = i\oint_{2C} \frac{\langle\tilde{\psi}_n(X)|d\psi_n(X)\rangle}{\langle\tilde{\psi}_n(X)|\psi_n(X)\rangle}, \tag{9}$$

where $\langle\tilde{\psi}_n(X)|$ is the left eigenvector corresponding to $E_n(X)$. Since the interacting states interchange after a single cycle, the Berry phases for these states are equal to each other ($\gamma_n = \gamma_{n+1}$). Note that the denominator in (9) vanishes at X_{EP}.

In the neighborhood of X_{EP}, the eigenvectors are given by

$$\begin{aligned} |\psi_{n,n+1}(X)\rangle &= |\chi_0^{\mathrm{EP}}\rangle \pm \sqrt{\mu\varepsilon}|\chi_1^{\mathrm{EP}}\rangle + o\left(\varepsilon^{\frac{1}{2}}\right), \\ \langle\tilde{\psi}_{n,n+1}(X)| &= \langle\tilde{\chi}_0^{\mathrm{EP}}| \pm \sqrt{\mu\varepsilon}\langle\tilde{\chi}_1^{\mathrm{EP}}| + o\left(\varepsilon^{\frac{1}{2}}\right) \end{aligned} \tag{10}$$

(see [8, 9]). Here,

$$\mu(X) = \langle\tilde{\chi}_0^{\mathrm{EP}}|H_1|\chi_0^{\mathrm{EP}}\rangle,\ H_1(X) = \sum_{k=1}^{m} \left.\frac{\partial H}{\partial X_k}\right|_{X=X_{\mathrm{EP}}} \hat{X}_k. \tag{11}$$

Substituting (10) and (11) into (9) and using the orthogonality and normalization conditions for $\langle\tilde{\chi}^{\mathrm{EP}}_{0,1}|$ and $|\chi^{\mathrm{EP}}_{0,1}\rangle$, we find

$$\gamma_n = \frac{i}{4}\oint_{2C} d\ln\mu + o(1) = \pi + o(1)(\mathrm{mod}\,2\pi). \quad (12)$$

The last equality follows from the logarithm rule, since the complex value μ goes around zero as the parameter vector makes a turn around the exceptional point.

For reversible systems, the Hamiltonian satisfies the condition $H^\dagger = \overline{H}$, where $\overline{H}$ is the complex conjugate operator (in the matrix version, this condition means the symmetry $H = H^T$). Then the left eigenvectors are equal to the complex conjugate right eigenvectors: $\langle\tilde{\psi}_n| = \overline{|\psi_n\rangle}$. In this case, the Berry phase is $\gamma_n = i\oint_{2C}\frac{d\ln\langle\overline{\psi_n(X)}\psi_n(X)\rangle}{2}$; i.e., it is determined only by the topology of the cycle relative to the degeneracies. Thus, expression (12) is exact: $\gamma_n = \pi$ (mod 2π). This situation differs from the case of real Hermitian Hamiltonians only in that the cycle C has to be traversed twice. We can say that, for reversible systems, the Berry phase around an exceptional point is half less than that around a diabolic point (per cycle traversed in the parameter space).

In the general case of irreversible non-Hermitian systems ($H^\dagger \neq \overline{H}$ and $H \neq H^\dagger$), the Berry phase depends on the cycle. To find the leading asymptotic term of this dependence, we represent the eigenvectors as

$$\begin{aligned}|\psi_{n,n+1}(X)\rangle &= |\chi_0(X)\rangle \pm \sqrt{p(X)}|\chi_1(X)\rangle,\\ \langle\tilde{\psi}_{n,n+1}(X)| &= \langle\tilde{\chi}_0(X)| \pm \sqrt{p(X)}\langle\tilde{\chi}_1(X)|.\end{aligned} \quad (13)$$

Here, $p(X) = \varepsilon\mu(X) + o(\varepsilon)$ is a smooth function of X. The vectors $|\chi_{0,1}(X)\rangle$ and $\langle\tilde{\chi}_{0,1}(X)|$ are smooth functions of X and satisfy the conditions

$$\begin{aligned}|\chi_{0,1}(X_{\mathrm{EP}})\rangle &= |\chi^{\mathrm{EP}}_{0,1}\rangle, \quad \langle\tilde{\chi}_{0,1}(X_{\mathrm{EP}})| = \langle\tilde{\chi}^{\mathrm{EP}}_{0,1}|,\\ \langle\tilde{\chi}_0(X)|\chi_0(X)\rangle &= \langle\tilde{\chi}_1(X)|\chi_1(X)\rangle = 0,\\ \langle\tilde{\chi}_1(X)|\chi_0(X)\rangle &= \langle\tilde{\chi}_0(X)|\chi_1(X)\rangle = 1.\end{aligned} \quad (14)$$

Formulas for the derivatives of $|\chi_{0,1}(X)\rangle$ and $\langle\tilde{\chi}_{0,1}(X)|$ with respect to the parameters at the point X_{EP} can be found by applying the theory of versal deformations [10]. As a result, combining (13) and (14) with (9), we find

$$\gamma_n = \pm\pi + ia\varepsilon^2 + o(\varepsilon^2), \quad (15)$$

where a is the constant defined by

$$a = \oint_C (2\langle\tilde{\chi}^{\mathrm{EP}}_0|H_1(G^{-3} - |\chi^{\mathrm{EP}}_1\rangle\langle\tilde{\chi}^{\mathrm{EP}}_1|)dH_1|\chi^{\mathrm{EP}}_0\rangle + \langle\tilde{\chi}^{\mathrm{EP}}_0|H_1G^{-2}dH_1|\chi^{\mathrm{EP}}_1\rangle + \langle\tilde{\chi}^{\mathrm{EP}}_1|H_1G^{-2}dH_1|\chi^{\mathrm{EP}}_0\rangle). \quad (16)$$

Here, $dH_1 = \sum_{k=1}^{m}\frac{\partial H}{\partial X_k}\Big|_{X=X_{\mathrm{EP}}} d\hat{X}_k$ and $G = H(X_{\mathrm{EP}}) - E_{\mathrm{EP}}I + |\chi^{\mathrm{EP}}_1\rangle\langle\tilde{\chi}^{\mathrm{EP}}_1|$, where I is the identity operator. The quantity $ia\varepsilon^2$ determines the leading term in the dependence of the Berry phase on the size and shape of C. It is found only from information on the system at the exceptional point X_{EP} (this information includes the eigen- and associated vectors and the first derivatives of the Hamiltonian with respect to the parameters).

Formula (16) can be transformed by using spectral expansions of the identity operator and the Hamiltonian (for simplicity, it is assumed that the Hamiltonian has a discrete spectrum):

$$I = |\chi^{\mathrm{EP}}_0\rangle\langle\tilde{\chi}^{\mathrm{EP}}_1| + |\chi^{\mathrm{EP}}_1\rangle\langle\tilde{\chi}^{\mathrm{EP}}_0| + \sum_{k\neq n,n+1}|\psi^{\mathrm{EP}}_k\rangle\langle\tilde{\psi}^{\mathrm{EP}}_k|, \quad (17)$$

$$H(X_{\mathrm{EP}}) = |\chi^{\mathrm{EP}}_0\rangle\langle\tilde{\chi}^{\mathrm{EP}}_0| + E_{\mathrm{EP}}(|\chi^{\mathrm{EP}}_0\rangle\langle\tilde{\chi}^{\mathrm{EP}}_1| + |\chi^{\mathrm{EP}}_1\rangle\langle\tilde{\chi}^{\mathrm{EP}}_0|) + \sum_{k\neq n,n+1}E^{\mathrm{EP}}_k|\psi^{\mathrm{EP}}_k\rangle\langle\tilde{\psi}^{\mathrm{EP}}_k|, \quad (18)$$

where $E^{\mathrm{EP}}_k = E_k(X_{\mathrm{EP}})$, $|\psi^{\mathrm{EP}}_k\rangle = |\psi_k(X_{\mathrm{EP}})\rangle$, and $\langle\tilde{\psi}^{\mathrm{EP}}_k| = \langle\tilde{\psi}_k(X_{\mathrm{EP}})|$. The right and left eigenvectors are assumed to be normalized: $\langle\tilde{\psi}^{\mathrm{EP}}_k|\psi^{\mathrm{EP}}_k\rangle = 1$. Note that $\langle\tilde{\psi}^{\mathrm{EP}}_k|\chi^{\mathrm{EP}}_{0,1}\rangle = \langle\tilde{\chi}^{\mathrm{EP}}_{0,1}|\psi^{\mathrm{EP}}_k\rangle = \langle\tilde{\psi}^{\mathrm{EP}}_{k'}|\psi^{\mathrm{EP}}_k\rangle = 0$ for $k \neq k'$. By substituting (17) and (18) into the expression for G, after some manipulations, expression (16) becomes

$$a = \sum_{k\neq n,n+1}\oint_C\left(2\frac{\langle\tilde{\chi}^{\mathrm{EP}}_0|H_1|\psi^{\mathrm{EP}}_k\rangle\langle\tilde{\psi}^{\mathrm{EP}}_k|dH_1|\chi^{\mathrm{EP}}_0\rangle}{(E^{\mathrm{EP}}_k - E_{\mathrm{EP}})^3} + \frac{\langle\tilde{\chi}^{\mathrm{EP}}_1|H_1|\psi^{\mathrm{EP}}_k\rangle\langle\tilde{\psi}^{\mathrm{EP}}_k|dH_1|\chi^{\mathrm{EP}}_0\rangle}{(E^{\mathrm{EP}}_k - E_{\mathrm{EP}})^2} + \frac{\langle\tilde{\chi}^{\mathrm{EP}}_0|H_1|\psi^{\mathrm{EP}}_k\rangle\langle\tilde{\psi}^{\mathrm{EP}}_k|dH_1|\chi^{\mathrm{EP}}_1\rangle}{(E^{\mathrm{EP}}_k - E_{\mathrm{EP}})^2}\right). \quad (19)$$

This formula sheds light on the physical nature of the dependence of the Berry phase on the shape of the cycle in the parameter space. The terms $\langle\tilde{\chi}^{\mathrm{EP}}_{0,1}|H_1|\psi^{\mathrm{EP}}_k\rangle$ and $\langle\tilde{\psi}^{\mathrm{EP}}_k|dH_1|\chi^{\mathrm{EP}}_{0,1}\rangle$ describe the interaction of the degenerate level E_{EP} with the other levels E_k, where $k \neq n, n+1$. Thus, the change in the Berry phase with cycle variations in the parameter space is due to the

influence of the energy levels not involved in the X_{EP} degeneracy. If the difference $E_k^{EP} - E_{EP}$ is large, the influence of E_k is proportional to $(E_k^{EP} - E_{EP})^{-2}$ and can be neglected. However, if $E_k^{EP} - E_{EP}$ is small, the change in the Berry phase due to the interaction with E_k increases proportionally to $(E_k^{EP} - E_{EP})^{-3}$ and may be large. In the case $E_k^{EP} - E_{EP} \to 0$, i.e., near the triple degeneracy $E_n = E_{n+1} = E_k$, we have $a \to \infty$. Thus, triple degeneracies require a special analysis.

For Hamiltonians represented by 2×2 matrices, we have $a = 0$, since the sum in (19) does not contain any term. In this case, it can be shown that $\gamma_n = \pi$ irrespective of the shape or size of C. This result was obtained in [14], where it underlay the conclusion that $\gamma_n = \pi$ in the general case. We have shown above that this is not the case. Specifically, the Berry phase deviates from π since the system is multidimensional. This deviation cannot be captured within the framework of two-dimensional approximations. This fact and asymptotics (15), (16) have been verified numerically for a few Hamiltonians described by nonsymmetric non-Hermitian matrices of sizes 3×3 and 4×4.

3. Thus, we have derived general asymptotic formulas for the Berry phase around (double) stationary and quasi-stationary degeneracies. Note that geometric phases defined in the form of integrals (1) and (7) arise in the optics of anisotropic media and in nonlinear dynamics [4, 6]. In optics, H is a Hermitian or non-Hermitian matrix obtained from the dielectric tensor of an anisotropic medium. In the case of dynamic systems, H is a linearization of the evolution operator. Thus, our general results can be successfully applied to the analysis of the geometric phase in various physical systems.

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REFERENCES

1. M. V. Berry, Proc. R. Soc. London A **392**, 45–57 (1984).
2. A. Shapere and F. Wilczek, *Geometric Phases in Physics* (World Scientific, Singapore, 1989).
3. A. Bohm, A. Mostafazadeh, H. Koizumi, *et al.*, *The Geometric Phase in Quantum Systems* (Springer-Verlag, Berlin, 2003).
4. S. I. Vinitskii, V. L. Derbov, V. M. Dubovik, *et al.*, Usp. Fiz. Nauk **160** (6), 1–49 (1990).
5. J. Anandan, J. Christian, and K. Wanelik, Am. J. Phys. **65**, 180–185 (1997).
6. A. S. Landsberg, Phys. Rev. Lett. **69**, 865–868 (1992).
7. V. I. Arnold, V. V. Kozlov, and A. I. Neishtadt, *Encyclopedia of Mathematical Sciences*, Vol. 3: *Mathematical Aspects of Classical and Celestial Mechanics* (Springer-Verlag, Berlin, 1988; Editorial URSS, Moscow, 2002).
8. A. P. Seyranian and A. A. Mailybaev, *Multiparameter Stability Theory with Mechanical Applications* (World Scientific, Singapore, 2003).
9. A. P. Seyranian, O. N. Kirillov, and A. A. Mailybaev, J. Phys. A: Math. Gen. **38**, 1723–1740 (2005).
10. A. A. Mailybaev, Lin. Algebra Appl. **337**, 87–108 (2001).
11. V. I. Arnold, Selecta Math. New Ser. **1** (1), 1–19 (1995).
12. M. V. Berry and M. Wilkinson, Proc. R. Soc. London A **392**, 15–43 (1984).
13. A. I. Baz', Ya. B. Zel'dovich, and A. M. Perelomov, *Scattering, Reactions, and Decays in Nonrelativistic Quantum Mechanics* (Israel Program for Scientific Translations, Jerusalem, 1966; Nauka, Moscow, 1971).
14. W. D. Heiss and H. L. Harney, Eur. Phys. J. Ser. D **17**, 149–151 (2001).
15. J. C. Garrison and E. M. Wright, Phys. Lett. A **128**, 177–181 (1988).